

Matrices over local rings: stability and finite determinacy

Genrich Belitskii AND Dmitry Kerner

ABSTRACT. Consider matrices with entries in a local ring, $Mat(m, n, R)$, suppose a group G acts on $Mat(m, n, R)$. The matrix A is G -stable if its orbit GA contains an open neighborhood of A . For the coarse topology on R this gives the classical notion of stability, while for the Krull topology this gives finite determinacy.

We provide the general effective iff criterion for finite determinacy of matrices. Using it we obtain various detailed criteria for particular group actions. For example, for two-sided multiplications, $G = GL(m, R) \times GL(n, R)$, for two-sided multiplications together with the automorphisms of the ring, $GL(m, R) \times GL(n, R) \rtimes Aut(R)$, for congruence, etc.

The results are of Mather type: for some tuples (m, n, R, G) there are no finitely-determined matrices, for others - the non-finitely-determined matrices are extremely rare.

In some case we classify the stable matrices.

CONTENTS

| | |
|---|----|
| 1. Introduction | 1 |
| 2. Generalities | 4 |
| 3. Implications for particular groups and subspaces of $Mat(m, n, R)$ | 7 |
| 4. Stable types | 14 |
| 5. Appendix | 14 |
| References | 16 |

1. INTRODUCTION

1.1. Setup. Let \mathbb{K} be a field of zero characteristic. Let (R, \mathfrak{m}) be a (commutative, Noetherian) local ring over \mathbb{K} . (As the simplest case, one can consider regular rings, e.g. the rational functions regular at the origin, $\mathbb{K}[x_1, \dots, x_p]_{(\mathfrak{m})}$ or the formal power series, $\mathbb{K}[[x_1, \dots, x_p]]$. For $\mathbb{K} \subseteq \mathbb{C}$ one can consider converging power series, $\mathbb{K}\{\cdot\}$.)

Let $\dim(R) < \infty$ denote the Krull dimension of (R, \mathfrak{m}) . Geometrically, R is the local ring of the (algebraic/formal/analytic etc.) germ $Spec(R)$ and $\dim(R)$ is the maximal among the dimensions of the irreducible components of $Spec(R)$. Usually we assume $\dim(R) > 0$, i.e. R is not Artinian. Let p denote the "embedding dimension" of $Spec(R)$, i.e. the \mathbb{K} -dimension of the vector space $\mathfrak{m}/\mathfrak{m}^2$. (If R is regular then $p = \dim(R)$.)

Let $Mat(m, n, R)$ be the space of $m \times n$ matrices with entries in R , we always assume $m \leq n$ (otherwise one can take the transpose). Various groups act on this space, e.g.

- the left multiplications $G_l := GL(n, R)$, the right and two-sided multiplications $G_r := GL(m, R)$, $G_{lr} := G_l \times G_r$;
- the "change of coordinates", $Aut(R)$, and the corresponding semi-direct products, $\mathcal{G}_l := G_l \rtimes Aut(R)$, $\mathcal{G}_{lr} := G_{lr} \rtimes Aut(R)$. In this case we assume the ring R to be henselian.
- For square matrices, $m = n$, one can consider e.g. conjugation ($A \rightarrow UAU^{-1}$) or congruence ($A \rightarrow UAU^T$), for $U \in GL(m, R)$.

In this paper a group $G \curvearrowright Mat(m, n, R)$ is always assumed to be a pro-algebraic subgroup of \mathcal{G}_{lr} , i.e. for any projection $Mat(m, n, R) \xrightarrow{jet_k} Mat(m, n, R/\mathfrak{m}^{k+1})$ the image $jet_k(G)$ is an algebraic subgroup of $jet_k(\mathcal{G}_{lr})$.

Convention. Recall that any matrix is G_{lr} equivalent to a block-diagonal, $A \stackrel{G_{lr}}{\sim} \mathbb{1} \oplus A'$, where all the entries of A' lie in the maximal ideal \mathfrak{m} , i.e. vanish at the origin. This splitting is preserved in deformation, i.e. given a family of matrices, $A_\epsilon \in Mat(m, n, R[[\epsilon]])$, with A_0 as above, this family is G_{lr} equivalent to $\mathbb{1} \oplus A'_\epsilon$, with $A'_\epsilon \in Mat(*, *, \mathfrak{m}[[\epsilon]])$. Therefore, in this work we mostly assume $A|_0 = \mathbb{0}$, i.e. $A \in Mat(m, n, \mathfrak{m})$.

1.2. Stability and finite determinacy. In this work we address the following classical question. Fix some $G \curvearrowright Mat(m, n, \mathfrak{m})$

For a "small" deformation $A \rightarrow A + B$, are the initial and deformed matrices G -equivalent?

Date: January 1, 2013.

2000 *Mathematics Subject Classification.* Primary 58K40, 58K50 Secondary 32A19, 14B07, 15A21.

Key words and phrases. Maps of Singular Germs, Matrix Singularities, Matrix Families, Finite Determinacy, Modules over Local Rings, Algebraization, Deformation of Modules.

Many thanks to V.Goryunov.

The answer depends essentially on the notion of "smallness". If the topology \mathcal{T}_1 (on $Mat(m, n, R)$) is coarser than \mathcal{T}_2 then \mathcal{T}_1 -stability implies \mathcal{T}_2 -stability. Thus it is natural to study stability for the coarsest and the finest topologies. We consider two extremal cases.

1.2.1. The strongest: ("classical") stability. Consider the projection $Mat(m, n, \mathfrak{m}) \xrightarrow{jet_1} Mat(m, n, \mathfrak{m}/\mathfrak{m}^2)$. Choose some (minimal) set of generators of $\mathfrak{m}/\mathfrak{m}^2$, identify $Mat(m, n, \mathfrak{m}/\mathfrak{m}^2)$ with the affine space \mathbb{K}^{mnp} . Then the Zariski topology on \mathbb{K}^{mnp} defines the one on $Mat(m, n, \mathfrak{m}/\mathfrak{m}^2)$. Consider the corresponding topology on $Mat(m, n, \mathfrak{m})$, the open sets being the preimages (under jet_1) of those in $Mat(m, n, \mathfrak{m}/\mathfrak{m}^2)$. Note that this topology is *coarse*, its open sets are huge, being cylinders over the large opens of $Mat(m, n, \mathfrak{m}/\mathfrak{m}^2)$. Call the deformation $A \rightarrow A + B$ "small" if B belongs to some (small enough) open neighborhood of the zero matrix \mathbb{O} . A matrix A is called *G*-stable if $A \stackrel{G}{\sim} A + B$ for any small B , alternatively, A is stable if its G -orbit contains an open neighborhood of A in $Mat(m, n, \mathfrak{m})$.

1.2.2. The weakest: finite determinacy. The maximal ideal of R induces the natural filtration:

$$(1) \quad Mat(m, n, R) \supset Mat(m, n, \mathfrak{m}) \supset Mat(m, n, \mathfrak{m}^2) \supset \dots$$

So, we have the Krull topology, generated by all the sets of the form $A + Mat(m, n, \mathfrak{m}^j)$. A matrix is called *k*-determined if $A \stackrel{G}{\sim} A + B$ for any $B \in Mat(m, n, \mathfrak{m}^{k+1})$. A matrix is called *finitely-G-determined* if it is k -determined for some $k \in \mathbb{N}$. (Alternatively, the orbit GA contains an open neighborhood of A , in Krull topology.)

Note that the Krull topology is much finer than the coarse topology (as above), in particular the (classical) stability implies one-determinacy and is close to the zero-determinacy. (Recall that we consider only matrices vanishing at the origin.)

1.3. Remarks.

1.3.1. Base Change. It is natural to relate the order of determinacy over a given ring R to that over some related ring. (For example, for $R = S/I$, with S -regular.) The expected relation exists if G does not involve the change of coordinates, but breaks completely otherwise (see §2.2 for the behavior of determinacy under base change).

1.3.2. Adjustments by higher orders. For any group $G \curvearrowright Mat(m, n, R)$ and some fixed $k \in \mathbb{N}$, consider the projection $R \xrightarrow{jet_k} R/\mathfrak{m}^{k+1}$. Then jet_k maps $Mat(m, n, R)$ to $Mat(m, n, R/\mathfrak{m}^{k+1})$ and G acts on $Mat(m, n, R/\mathfrak{m}^{k+1})$ through the corresponding group $jet_k(G)$. Consider $G^{(k)} := (jet_k)^{-1}(e) \subset G$, i.e. all the elements of G that act trivially on $Mat(m, n, R/\mathfrak{m}^{k+1})$, i.e. transformations that are identities "up to k 'th order". The finite determinacy of A is established by studying the tangent space to the orbit, $T_{(GA, A)}$, §2.3. As the inclusion $T_{(G^{(k)}A, A)} \subset T_{(GA, A)}$ is of finite codimension, we get: A is finitely- G -determined iff it is finitely- $G^{(k)}$ -determined.

1.3.3. Smooth and analytic matrices. In most of the paper R is assumed to be Noetherian, in particular this excludes the rings of smooth functions. However, our results are applicable to C^∞ category. We prove, in §5.1, that a C^∞ matrix is finitely- G -determined iff its completion is finitely-determined (for the completion of G).

Similarly, for $\mathbb{K} \subset \mathbb{C}$, an analytic matrix is finitely (analytically) determined iff it is finitely formally determined.

1.3.4. In the particular case, $R = S[x_1, \dots, x_p]/(x_i x_j)$, $G = G_{lr}$, we have the p -tuples of matrices over S , i.e. the representation theory of the quiver with two vertices and p arrows.

1.3.5. Formulation in terms of maps. Ignoring the matrix structure we can consider $Mat(m, n, R)$ as the space of maps, $Maps(Spec(R), (\mathbb{K}^{mn}, 0))$. The contact group, \mathcal{K} , acts on this space and $\mathcal{G}_{lr} \subset \mathcal{K}$. (This inclusion is "tight", i.e. \mathcal{G}_{lr} cannot be enlarged further, §2.1.) Thus the finite- \mathcal{G} -determinacy of a matrix implies the finite- \mathcal{K} -determinacy of the corresponding map. In the inverse direction, starting from the space $Maps(Spec(R), (\mathbb{K}^N, 0))$, with $N = mn$, we can associate to any map the corresponding matrix. As the orbits of the group \mathcal{G}_{lr} are much smaller than those of \mathcal{K} , we get a much stronger notion of finite determinacy.

1.3.6. Restricting to subsets of $Mat(m, n, R)$. Recall that congruence, $A \rightarrow UAU^T$, preserves the (anti)symmetry. Therefore no (anti)symmetric matrix can be finitely determined in $Mat(m, n, R)$ for $G_{congr} \rtimes Aut(R)$. However, in this case it is natural to consider only deformations inside the *subspaces* of (anti)symmetric matrices, §3.3.

More generally, for many groups $G \curvearrowright Mat(m, n, \mathfrak{m})$, there are no finitely determined matrices, for most rings. Thus, it is natural to restrict to some subspaces of $Mat(m, n, \mathfrak{m})$, for which one expects generic finite- G -determinacy. The general results of §2.3 apply also for subspaces of $Mat(m, n, \mathfrak{m})$, defined by finite sets of algebraic conditions, admitting the R -multiplication, which are also free R -modules. (For example, (anti)symmetric matrices, upper/lower triangular, with zero trace etc.)

We note, that in all the examples considered, the finite determinacy criterion for $G \subset G_{lr}$ is written in terms of the fitting ideal $I_m(A)$ or $I_{m-1}(A)$. It will be very interesting to obtain some general statement of the type:

Let $M_0 \subset Mat(m, n, \mathbb{K})$ be a vector subspace, consider the corresponding R module $M := jet_0^{-1}(M_0)$. Let $G \curvearrowright Mat(m, n, \mathfrak{m})$ be the maximal group that acts on M , suppose $G \subset G_{lr}$. Then the finite determinacy of $A \in M$ is determined by $I_m(A)$ or $I_{m-1}(A)$ only.

1.4. Results. In §2.3 we obtain the general criterion of finite determinacy, for a given (m, n, R, G) , in terms of the ideal of maximal minors of some associated matrix \mathcal{A} . The power of this criterion is shown by application to particular groups, to G_{lr} and G_r in §3.1, to $Aut(R)$ in §3.7, to G_{congr} for (anti)symmetric matrices in §3.3. In some cases we classify the stable matrices too. In the appendix we relate the results to the smooth/analytic categories.

1.5. Relations to other fields and motivation.

1.5.1. Singularity Theory. The study of stability/simple types/finite determinacy is the usual starting point, [AGLV-1], [AGLV-2], [Looijenga-book], [du Plessis-Wall], for $\mathbb{R} \subseteq \mathbb{K} \subseteq \mathbb{C}$ and $R = \mathbb{K}\{x_1, \dots, x_p\}$ or $\mathbb{K}[[x_1, \dots, x_p]]$ or $C^\infty(\mathbb{K}^p, 0)$. The criteria are usually formulated over the algebraic closure $\mathbb{K} \subset \bar{\mathbb{K}}$.

- For functions, $m = 1 = n$, the simple singularities are the ADE singularities, while the finite determinacy means that the singularity of $f^{-1}(0) \subset (\bar{\mathbb{K}}^p, 0)$ is isolated.
- For $m = 1$ and the regular ring R we have $maps, Maps((\bar{\mathbb{K}}^p, 0), (\bar{\mathbb{K}}^n, 0))$. The equivalence induced by \mathcal{G}_{lr} coincides with the contact equivalence, \mathcal{K} . A map $F \in Maps((\bar{\mathbb{K}}^p, 0), (\bar{\mathbb{K}}^n, 0))$ is finitely determined iff either $F^{-1}(0) \subset (\bar{\mathbb{K}}^p, 0)$ is a zero-dimensional scheme or $F^{-1}(0)$ is a complete intersection (of expected dimension $(p - n)$) with an isolated singularity. In particular, the *generic* map is finitely \mathcal{G}_{lr} -determined. (Here the genericity is taken in the sense of [Tougeron1968]: the subset of non-finitely determined maps is of infinite codimension in $Maps((\bar{\mathbb{K}}^p, 0), (\bar{\mathbb{K}}^n, 0))$.) There are additional simple types, as compared to the case of functions (i.e. $n = 1$).
- The case of square matrices (for $\mathbb{R} \subseteq \mathbb{K} \subseteq \mathbb{C}$, $R = \mathbb{K}\{x_1, \dots, x_p\}$ and $G = \mathcal{G}_{lr}$) was considered in [Bruce-Tari2004], and further studied in [Bruce-Goryunov-Zakalyukin2002], [Bruce03], [Goryunov-Mond05], [Goryunov-Zakalyukin03]. In particular, the generic finite determinacy was established and the simple types were classified.
- Finite determinacy is equivalent to the finite dimensionality of the miniversal deformation. In particular, the genericity of finite determinacy, for a fixed (m, n, R, G) , means: the stratum Σ_∞ of germs of Tjurina number ∞ is of infinite codimension in $Mat(m, n, R)$.

Remark 1.1. Note that we consider *local* situation: (R, \mathfrak{m}) is a local ring and $Spec(R)$ is the (algebraic/formal/analytic etc.) germ at the origin. Thus any change of coordinates *preserves the origin*, i.e. for any $\phi \in Aut(R)$: $\phi(\mathfrak{m}) = \mathfrak{m}$. So, e.g. the Morse critical points, $f = \sum_i x_i^2$, are not stable in our approach.

1.5.2. Relation to Commutative Algebra. For the relevant background cf. [Eisenbud-book].

- Any matrix is the resolution matrix of its cokernel, a module over R , $R^{\oplus n} \xrightarrow{A} R^{\oplus m} \rightarrow coker(A) \rightarrow 0$. From commutative algebra point of view the classical stability implies the rigidity of the module, while finite determinacy means that the miniversal deformation of a module is of finite dimension.

For $0 \leq j \leq m$ and $A \in Mat(m, n, R)$ let $I_j(A) \subset R$ be the *fitting* ideal, generated by all the $j \times j$ minors of A . (By definition $I_0(A) = R$.) Note that the chain of ideals $R = I_0(A) \supsetneq I_1(A) \supsetneq \dots \supsetneq I_m(A)$ is *invariant* under G_{lr} action and admits the action of $Aut(R)$.

Note that the module is completely determined by the restriction onto its support, $Supp(coker(A)) = V(I_m(A))$. In terms of matrices this means: $A \xrightarrow{\mathcal{G}_{lr}} B$ iff $A \xrightarrow{\mathcal{G}_{lr}} B \bmod(I_m(A))$. As the dimension of the ring $R/I_m(A)$ is usually smaller than $dim(R)$, it is often helpful to restrict onto the support of the module.

A particularly important (and well studied) case is: $m = n$ and $\det(A) \neq 0$. Then A is a presentation matrix of a *maximally Cohen-Macaulay* module $coker(A)$, [Yoshino-book], [Leuschke-Wiegand].

- The projection $R \xrightarrow{jet_{k-1}} R/\mathfrak{m}^k$ induces the map of categories $Mod_R \rightarrow Mod_{R/\mathfrak{m}^k}$, defined by $M \rightarrow M/\mathfrak{m}^k M$. This map is surjective, an R/\mathfrak{m}^k module M is also an R module, for $r \in R$ define $rM := jet_{k-1}(r)M$. But the map is usually not injective, e.g. if the presentation matrix of M_R has all its entries in \mathfrak{m}^k then the image is a free R/\mathfrak{m}^k module. Our results give the regions of parameters (the size of presentation matrix, the dimension and embedded dimension of the ring) for which such map is "generically injective".

- The general line of research is: "which information about a module is determined by its fitting ideals?" In particular, do the properties of stability/finite determinacy involve essentially the properties of modules or only of their fitting ideals? As our results show, if G does not involve the change of coordinates, then the finite- G -determinacy is a property of the zeroth fitting ideal of a module. Another classical question is: suppose for the two modules over R the (corresponding) fitting ideals coincide. What are the additional conditions to ensure that the modules are isomorphic (i.e. their resolution matrices are G_{lr} equivalent)? Again, finite determinacy implies: we should check only the relevant fitting ideals and to compare the modules over R/\mathfrak{m}^N , for some large N .

1.5.3. Relation to the Algebraization Problem. If a matrix A is k -determined then, in particular, it is G -equivalent to a matrix whose entries are polynomials of degrees at most k . Therefore finite determinacy is a (significant) strengthening of the old problem of algebraization: "which objects have polynomial representatives?" More precisely, there are two versions. Suppose R is the localization/henselization/completion of S , with the natural embedding $S \hookrightarrow R$. Consider a (finitely generated) module M_R .

★ When does there exist a (finitely generated module) N_S satisfying $M_R = R \otimes_S N_S$? (Geometrically, given a fixed affine scheme X , consider its algebraic/analytic/formal germ at a point, $(X, 0)$. Which modules over $(X, 0)$ come as the stalks

of some sheaves over X ?)

★ When can S be re-embedded, $S \xrightarrow{\phi} R$, such that M_R comes as the extension, $M_R = R \otimes_{\phi(S)} N_S$? (Geometrically, given some analytic/formal germ $(X, 0)$, which modules over $(X, 0)$ can be realized as the stalks of some sheaves on some affine scheme, whose germ is $(X, 0)$?)

The first version is the weakening of finite- G_{lr} -determinacy, the second is the weakening of finite- \mathcal{G}_{lr} -determinacy.

Therefore our results imply, in particular, various extensions and strengthening of the classical algebraization criteria for modules over local rings ([Elkik73], for many additional results and bibliography cf. [Christensen-Sather-Wagstaff]). We mention briefly two the recent developments.

• [FSWW08]:

Proposition 3.3: *Let (R, \mathfrak{m}) be a local ring, $\dim(R) = 1$, whose completion \hat{R} is a domain. Every finitely generated module over \hat{R} is the extension of a module over R .*

Theorem 3.4: *Let $S \xrightarrow{\phi} R$ be a flat local homomorphism, assume R is separable over S . Then every finitely generated module over R is a direct summand of an extension of a module from S .*

Corollary 3.5: *In particular, for henselization, $R \rightarrow R^h$, every finitely generated module over R^h is a direct summand of an extension of a module from R .*

Example 3.6 of their paper shows that this property fails for completions.

• Much of research has been done for maximally Cohen-Macaulay modules (mCM), these are the modules whose presentation matrix (A in our case) is square. In [Keller-Murfet-Van den Bergh2008] they prove: if (R, \mathfrak{m}) is a Gorenstein local ring, whose completion, (\hat{R}, \mathfrak{m}) , has isolated singularities, then every mCM module over \hat{R} is a direct summand in the completion of a module over R .

We remark that nothing of that type can hold for finite- G_{lr} -determinacy, in view of our theorem 3.1 (for the direct sum the maximal fitting ideal of the matrix is the product, thus the direct sum of modules is finitely- G_{lr} -determined iff each of them is).

• Closely related are "Maranda type results" in commutative algebra, [Leuschke-Wiegand, §15.2]. Let (R, \mathfrak{m}) be a Cohen-Macaulay ring admitting a faithful system of parameters, $x_1, \dots, x_{\dim(R)}$.

Corollary 15.9. *If for two mCM modules M, N : $M/(\{x_i^2\})M \xrightarrow{\phi} N/(\{x_i^2\})N$, then there exists an isomorphism $M \xrightarrow{\tilde{\phi}} N$ such that $\tilde{\phi} \otimes_R R/(\{x_i\}_i) = \phi \otimes_R R/(\{x_i\}_i)$*

(The faithful system of parameters exists e.g. for a complete Cohen-Macaulay ring with isolated singularity.)

In terms of presentation matrices, A_M, A_N , this implies: if $\det(A_M) = \det(A_N) = 0 \in R$ and $A_M \otimes R/(\mathbf{x}^2) \xrightarrow{G_{lr}} A_N \otimes R/(\mathbf{x}^2)$, then $A_M \xrightarrow{G_{lr}} A_N$.

2. GENERALITIES

2.1. Groups acting on matrices. When studying stability/finite determinacy, it is natural to start from the biggest possible groups, i.e. the weakest (reasonable) equivalence, even if it violates the matrix structure. In the case of maps, the contact equivalence (§1.5.1) is satisfactory in various senses. Note that we can consider $\text{Maps}((\mathbb{K}^p, 0), (\mathbb{K}^n, 0))$ as $\text{Mat}(1, n, S)$, (here S is the local ring of $(\mathbb{K}^p, 0)$, e.g. algebraic functions or formal power series) then the contact equivalence is induced by the action of \mathcal{G}_{lr} .

Similarly, we can consider matrices over R as maps from $\text{Spec}(R)$ to $\text{Mat}(m, n, \mathbb{K})$, i.e. consider $\text{Mat}(m, n, R)$ as $\text{Maps}(\text{Spec}(R), (\mathbb{K}^{mn}, 0))$. Then the contact equivalence is induced by $GL(mn, R) \rtimes \text{Aut}(R)$. Therefore, it is natural to consider only those groups, $G \subset \text{Mat}(m, n, R)$, that are subgroups of $GL(mn, R) \rtimes \text{Aut}(R)$. Besides, the equivalence should, at least, distinguish between degenerate and non-degenerate matrices (i.e. matrices of distinct ranks). Groups with such properties are restricted by the following classical result.

Theorem 2.1. [Dieudonné1949] *Let T be an invertible map acting linearly on the vector space $\text{Mat}(m, m, \mathbb{K})$ (possibly violating the matrix structure). Suppose T acts on the set of degenerate matrices, i.e. $\det(A) = 0$ iff $\det(TA) = 0$. Then either $T(A) = UAV$ or $T(A) = UA^T V$, for some $U, V \in GL(m, \mathbb{K})$.*

(For the general introduction to the theory of preservers, i.e. self-maps of $\text{Mat}(m, m, \mathbb{K})$, that preserve some properties/structures cf. [Molnár2007].)

Therefore, in this paper G is always a (closed) subgroup of \mathcal{G}_{lr} .

2.2. Change of base ring. If the group does not involve the change of coordinates, i.e. $G \subseteq G_{lr}$, then the order of determinacy behaves well under the change of base ring.

Proposition 2.2. *Fix some $G \subseteq G_{lr}$.*

1. *If $R \rightarrow S \rightarrow 0$ then the order of determinacy over R is bigger or equal to that over S .*

2. Consider a homomorphism $R \xrightarrow{\phi} S$, suppose $\dim_{\mathbb{K}}(S/\phi(R)) < \infty$. If A_R is k -determined with $k \geq \dim_{\mathbb{K}}(S/\phi(R))$, then $A_R \otimes_{\phi(R)} S$ is k -determined.

Thus, if $G \subseteq G_{lr}$, i.e. does not involve coordinate changes, then the stratum of not- G -finitely determined matrices, $\Sigma_{\infty} \subset \text{Mat}(m, n, \mathfrak{m})$, behaves well under the base-change (cf. remark 2.7).

But for equivalence involving changes of coordinates the situation is much worse, e.g. finite determinacy over R neither implies nor is implied by that over R/I (for some ideal I).

Example 2.3. The function $f = x + y^2 \in \mathbb{K}[[x, y, z]]$ is finitely determined (even stable!) though its image in $\mathbb{K}[[x, y, z]]/(x)$ is not finitely determined. (Alternatively, the image of f in $R/(f)$ is just zero.) In the other direction, suppose $\dim(R) = 2$, let $f \in \mathfrak{m}$ (i.e. f vanishes at the origin). Then $f^2 \in R$ is not finitely- \mathcal{K} -determined. But for any (generic enough) ideal $J \subset R$ such that the ideal $(f, J) \subset R$ is of height two (i.e. the two germs intersect at the origin only) the element $f^2 \in R/J$ is finitely determined.

Let $\text{Nilp}(R)$ be the ideal of all the nilpotent elements, so the ring $R/\text{Nilp}(R)$ is reduced.

Lemma 2.4. For any $G \subseteq G_{lr}$, finite determinacy over R is equivalent to that over $R_{red} := R/\text{Nilp}(R)$.

Proof. The direct statement is trivial. For the converse statement, choose some $k \in \mathbb{N}$ be such that $\text{Nilp}(R)^k = \{0\} \subset R$ and use lemma 2.5. Then, if $I_{max}(\mathcal{A} \otimes R/\text{Nilp}(R)) \supset \mathfrak{m}_{R_{red}}^N$ it follows that $I_{max}(\mathcal{A}) \supset \mathfrak{m}_R^{Nk}$. So, by the same lemma, A is finitely- G -determined. ■

2.3. Tangent space to the orbit and its presentation matrix. Let $G \odot \text{Mat}(m, n, \mathfrak{m})$, let $T_{(GA,A)}$ be the tangent space to the orbit of G at $A \in \text{Mat}(m, n, \mathfrak{m})$. This tangent space is naturally embedded into $T_{(\text{Mat}(m, n, \mathfrak{m}), A)} \approx \text{Mat}(m, n, \mathfrak{m})$. The classical criteria of Singularity Theory read:

- The matrix $A \in \text{Mat}(m, n, \mathfrak{m})$ is G -stable iff this embedding is an isomorphism.
- The matrix is G -finitely determined iff the embedding $T_{(GA,A)} \hookrightarrow \text{Mat}(m, n, \mathfrak{m})$ is of finite codimension, i.e.

$$\dim_{\mathbb{K}} \left(\text{Mat}(m, n, \mathfrak{m}) / T_{(GA,A)} \right) < \infty.$$

2.3.1. The criterion in terms of the fitting ideal. Let \mathcal{A} be a generating matrix of $T_{(GA,A)}$, i.e. $T_{(GA,A)}$ is the image of the map $M \xrightarrow{\mathcal{A}} \text{Mat}(m, n, R)$. (Here M is some free R -module, $\text{Mat}(m, n, R)$ is also considered as a free module, of rank $= mn$, so \mathcal{A} is of size $mn \times \text{rank}(M)$.) We assume the presentation to be minimal (i.e. $\text{rank}(M)$ to be minimal possible), so the columns of \mathcal{A} are linearly independent over \mathbb{K} . Further, as $T_{(GA,A)} \subset \text{Mat}(m, n, \mathfrak{m})$, all the entries of \mathcal{A} lie in \mathfrak{m} .

Lemma 2.5. 1. $A \in \text{Mat}(m, n, \mathfrak{m})$ is finitely- G -determined iff $I_{max}(\mathcal{A})$ contains a power of the maximal ideal of R .
2. If $A \in \text{Mat}(m, n, \mathfrak{m})$ is G -stable then $I_{max}(\mathcal{A}) = \mathfrak{m}^{mn} \subset R$. If R is a regular ring, then the converse holds too.

Proof. Consider the cokernel, $M \xrightarrow{\mathcal{A}} \text{Mat}(m, n, R) \rightarrow \text{coker}(\mathcal{A}) \rightarrow 0$, it is an R -module.

1. Finite determinacy means the finiteness of the codimension $T_{(GA,A)} \subset \text{Mat}(m, n, R)$, which is equivalent to the finite dimensionality of the module $\text{coker}(\mathcal{A})$ (as K -vector space). And this is equivalent to: the annihilator of $\text{coker}(\mathcal{A})$ contains a power of the maximal ideal. But, by [Eisenbud-book, pg.498] the radicals of $\text{Ann}(\text{coker}(\mathcal{A}))$ and $I_{max}(\mathcal{A})$ coincide. Hence the first statement.

2. As $\mathcal{A} \in \text{Mat}(mn, *, \mathfrak{m})$ we get $I_{max}(\mathcal{A}) \subseteq \mathfrak{m}^{mn}$. If A is stable then $T_{(GA,A)} = \text{Mat}(m, n, \mathfrak{m})$, so $\text{Ann}(\text{coker}(\mathcal{A})) = \mathfrak{m}$. Recall, [Eisenbud-book, proposition 20.7], that $I_{max}(\mathcal{A}) \subseteq \text{Ann}(\text{coker}(\mathcal{A}))$ and if $\text{coker}(\mathcal{A})$ can be generated by k elements then $(\text{Ann}(\text{coker}(\mathcal{A})))^k \subseteq I_{max}(\mathcal{A})$. In our case, $\text{coker}(\mathcal{A})$ is minimally generated by mn elements, thus $\mathfrak{m}^{mn} \subseteq I_{max}(\mathcal{A})$. Combining with the previous we get $I_{max}(\mathcal{A}) = \mathfrak{m}^{mn}$.

Vice-versa, suppose $I_{max}(\mathcal{A}) = \mathfrak{m}^{mn}$ then, as mentioned above, $\mathfrak{m}^{mn} \subseteq (\text{Ann}(\text{coker}(\mathcal{A})))^{mn}$ while, obviously, $\text{Ann}(\text{coker}(\mathcal{A})) \subseteq \mathfrak{m}$. Thus $(\text{Ann}(\text{coker}(\mathcal{A})))^{mn} = \mathfrak{m}^{mn}$. If R is regular, then this implies $\text{Ann}(\text{coker}(\mathcal{A})) = \mathfrak{m}$, i.e. $T_{(GA,A)} = \text{Mat}(m, n, \mathfrak{m})$, which is the stability. ■

Remark 2.6. In general, for non-regular ring, the condition $I_{max}(\mathcal{A}) = \mathfrak{m}^{mn}$ does not imply stability. For example, given any ring (S, \mathfrak{m}_S) , consider $R = S[\epsilon]/\{\epsilon^j \mathfrak{m}_S^{mn-j}\}_{j=1 \dots mn}$, then $\mathfrak{m}_R^{mn} = \mathfrak{m}_S^{mn}$. Suppose $A \in \text{Mat}(m, n, \mathfrak{m}_S)$ is stable, so $I_{max}(\mathcal{A}) = \mathfrak{m}_S^{mn} = \mathfrak{m}_R^{mn}$. But, of course, A is not stable inside $\text{Mat}(m, n, \mathfrak{m}_R)$.

Remark 2.7. Lemma 2.5 determines directly, for any action $G \odot \text{Mat}(m, n, R)$, the "worst" stratum $\Sigma_{\infty}^{\text{Mat}(m, n, R)}$ of not finitely-determined matrices. In many questions of geometry/singularity theory the "worst stratum" is defined as the last step of a stratification procedure, it is defined inductively, raising various associated questions of complexity. In our case, this worst stratum is defined by the explicitly written condition.

For $G \subseteq G_{lr}$ the definition of $\Sigma_{\infty}^{\text{Mat}(m, n, R)}$ is functorial under base change. Namely, for any homomorphism of rings $S \xrightarrow{\phi} R$ and the associated map $\text{Mat}(m, n, S) \xrightarrow{\phi} \text{Mat}(m, n, R)$, the stratum transforms: $\phi(\Sigma_{\infty}^{\text{Mat}(m, n, S)}) = \Sigma_{\infty}^{\text{Mat}(m, n, R)}$,

i.e. the ideal (of $\Sigma_{\infty}^{Mat(m,n,S)}$ in S) is mapped to the ideal (of $\Sigma_{\infty}^{Mat(m,n,R)}$ in R). Moreover, if ϕ is injective, then $\phi^*(\Sigma_{\infty}^{Mat(m,n,R)}) = (\Sigma_{\infty}^{Mat(m,n,S)})$.

Remark 2.8. As we show in theorem 3.1, the condition " $I_{max}(\mathcal{A})$ contains some power of \mathfrak{m} " is equivalent to: " \mathcal{A} is finitely G_r -determined". Similarly, for regular local rings, $I_{max}(\mathcal{A}) = \mathfrak{m}^{mn}$ iff A is G_r -stable. Therefore the lemma implies the general statement:

The stability/finite determinacy for an arbitrary group $G \subseteq \mathcal{G}_{lr}$ is reduced to those for G_r .

In other words, the problem for an arbitrary group is "contained" in the problem for G_r .

2.3.2. Criterion in terms of left kernel. For any given ring S , the left kernel of a matrix $B \in Mat(m, n, S)$ is defined by

$$(2) \quad Ker^{(l)}(B) := \{v \in S^{\oplus m} \mid vB = 0 \in S^{\oplus n}\}.$$

By construction, it is an S submodule of the free module $S^{\oplus m}$, thus it is torsion-free (or zero).

Lemma 2.9. *Given $A \in Mat(m, n, R)$ and a group G , let \mathcal{A} be the corresponding generating matrix of $T_{(GA,A)}$. Then A is finitely determined iff for any radical ideal $J \subset R$, that is not the maximal ideal, the matrix $\mathcal{A} \otimes R/J$ has no left kernel.*

Proof. By the general commutative algebra, the support of the left kernel is precisely the radical of $I_{max}(\mathcal{A})$, i.e. $\mathcal{A} \otimes R/J$ has a non-trivial left kernel iff $J \supseteq \sqrt{I_{max}(\mathcal{A})}$. Thus, if $I_{max}(\mathcal{A})$ contains a power of the maximal ideal, then for any radical ideal J that is not \mathfrak{m} , $J \not\supseteq I_{max}(\mathcal{A})$. Thus $\mathcal{A} \otimes R/J$ has no left kernel.

Suppose $I_{max}(\mathcal{A})$ does not contain any power of the maximal ideal, then the quotient ring $R/I_{max}(\mathcal{A})$ has a positive Krull dimension. Taking $J = \sqrt{I_{max}(\mathcal{A})}$, we get: $I_{max}(\mathcal{A} \otimes R/J) = 0 \in R$, thus $\mathcal{A} \otimes R/J$ has the non-zero left kernel. ■

Remark 2.10. Suppose the ring R is analytic or algebraic, then we can take a representative of the germ $Spec(R)$, a (small open) neighborhood of the origin, and this representative has closed points (besides the origin). Then, for any point in this neighborhood we can evaluate the matrix, $\mathcal{A}|_{pt}$, this is a *numerical* matrix. Then the condition, " $I_{max}(\mathcal{A})$ contains a power of the maximal ideal", is immediately translated into: for any point of the punctured neighborhood of the origin, $pt \in Spec(R) \setminus \{0\}$, there is a function $f \in I_{max}(\mathcal{A})$ with $f(pt) \neq 0$. Or, alternatively, the left kernel of the *numerical* matrix $\mathcal{A}|_{pt}$ is trivial. For formal rings, we cannot compute \mathcal{A} at a point off the origin. The statement of the lemma is just the reformulation of " $Ker^{(l)}(\mathcal{A}|_{pt}) = 0$ " condition to the case of an arbitrary ring.

Thus, to check the stability/finite determinacy, we should write down the matrix \mathcal{A}_G for a given group G .

2.4. The matrices \mathcal{A}_{G_r} , \mathcal{A}_{G_l} and $\mathcal{A}_{Aut(R)}$. Consider the (most inclusive) case \mathcal{G}_{lr} . To construct the tangent space, we consider the "infinitesimal" action of \mathcal{G}_{lr} : $A(\underline{x}) \rightarrow (\mathbb{I} + \epsilon U)A(\underline{x} + \epsilon\phi(\underline{x}))(\mathbb{I} + \epsilon V)$, for $\epsilon \in \mathbb{K}$ a parameter, where $\phi \in End(R)$, in particular $\phi(0) = 0$. So, $T_{(\mathcal{G}_{lr}, A, A)}$ is spanned by all the possible combinations of UA , AV , $\sum_{\nu} \phi_{\nu}(\underline{x})\partial_{\nu}A$. By choosing U, V as elementary matrices we get that $T_{(\mathcal{G}_{lr}, A, A)}$ is generated, as an R -module, by:

$$\left\{ \sum_j A_{ij} E_{kj} \right\}_{i,k}, \quad \left\{ \sum_i A_{ij} E_{ik} \right\}_{j,k}, \quad \{ \mathfrak{m} \mathcal{D} A \}_{\mathcal{D} \in Der(R)}$$

Here E_{ij} are elementary matrices, $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. In the first brackets we have matrices with only one non-zero row, an arbitrary row of A , in the second - matrices with only one non-zero columns, an arbitrary column of A . In the third brackets we have derivations \mathcal{D} of R applied to A . Denote the generators of R -module $Der(R)$ by $\partial_1, \dots, \partial_p$. (If R is a regular ring, then these are just the ordinary derivatives.) As for finite determinacy we need only the finiteness of the dimension of $Mat(m, n, \mathfrak{m})/T_{(GA,A)}$, we can consider the module generated by $\{\partial_i A\}$, instead of $\{\mathfrak{m} \partial_i A\}$, i.e. we consider the module generated by

$$\left\{ \sum_j A_{ij} E_{kj} \right\}_{i,k}, \quad \left\{ \sum_i A_{ij} E_{ik} \right\}_{j,k}, \quad \{ \partial_i A \}_{i=1..p}$$

Identify $Mat(m, n, R)$ with column vectors in $R^{\oplus mn}$. We present the matrix $\mathcal{A} \in Mat(mn, p + m^2 + n^2, R)$ in three blocks. The first $mn \times p$ block corresponds to the change of variables, i.e. $Aut(R)$, the second $mn \times n^2$ to G_r , the third

$mn \times m^2$ to G_l . In the basis $(a_{11}, a_{12}, \dots, a_{1n}, a_{21}, \dots, a_{2n}, \dots, a_{m1}, \dots, a_{mn})$ we have:

$$(3) \quad \mathcal{A} = \underbrace{\begin{pmatrix} \vec{\partial}a_{11} \\ \vec{\partial}a_{12} \\ \dots \\ \vec{\partial}a_{1n} \\ \vec{\partial}a_{21} \\ \vec{\partial}a_{22} \\ \dots \\ \vec{\partial}a_{2n} \\ \dots \\ \vec{\partial}a_{m1} \\ \dots \\ \vec{\partial}a_{mn} \end{pmatrix}}_{Aut(R)} \underbrace{\begin{pmatrix} a_{11} & \dots & a_{1n} & 0 & 0 & \dots & & & \\ 0 & \dots & 0 & a_{11} & \dots & a_{1n} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} & 0 & 0 & \dots & & & \\ 0 & \dots & 0 & a_{21} & \dots & a_{2n} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & a_{21} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} & 0 & 0 & \dots & & & \\ 0 & \dots & 0 & a_{m1} & \dots & a_{mn} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & a_{m1} & \dots & a_{mn} \end{pmatrix}}_{G_r} \underbrace{\begin{pmatrix} A^T & \mathbb{O} & \mathbb{O} & \dots \\ \mathbb{O} & A^T & \mathbb{O} & \dots \\ & \dots & \dots & \dots \\ \mathbb{O} & \dots & \dots & A^T \end{pmatrix}}_{G_l}$$

(Here $\vec{\partial}$ denotes the row of derivatives, $\vec{\partial}a = (\partial_1 a, \dots, \partial_p a)$.)

Sometimes we need this matrix in the basis $(a_{11}, a_{21}, \dots, a_{m1}, a_{12}, \dots, a_{m2}, \dots, a_{1n}, \dots, a_{mn})$:

$$(4) \quad \mathcal{A} = \underbrace{\begin{pmatrix} \vec{\partial}a_{11} \\ \dots \\ \vec{\partial}a_{m1} \\ \vec{\partial}a_{12} \\ \dots \\ \vec{\partial}a_{m2} \\ \dots \\ \vec{\partial}a_{mn} \end{pmatrix}}_{Aut(R)} \underbrace{\begin{pmatrix} A & \mathbb{O} & \mathbb{O} & \dots \\ \mathbb{O} & A & \mathbb{O} & \dots \\ & \dots & \dots & \dots \\ \mathbb{O} & \dots & \dots & A \end{pmatrix}}_{G_r} \underbrace{\begin{pmatrix} a_{11} & \dots & a_{m1} & 0 & 0 & \dots & & & \\ 0 & \dots & 0 & a_{11} & \dots & a_{m1} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & a_{11} & \dots & a_{m1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{1n} & \dots & a_{mn} & 0 & 0 & \dots & & & \\ 0 & \dots & 0 & a_{1n} & \dots & a_{mn} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & a_{1n} & \dots & a_{mn} \end{pmatrix}}_{G_l}$$

Denote by \mathcal{A}_{G_l} , \mathcal{A}_{G_r} , $\mathcal{A}_{G_{lr}}$, $\mathcal{A}_{Aut(R)}$ the corresponding blocks of \mathcal{A} .

Corollary 2.11. *A is finitely G-determined iff for any radical ideal $J \subsetneq \mathfrak{m}$ the left kernel module is trivial:*

$$G = G_{lr} : \quad Ker^{(l)}(\mathcal{A}_{G_{lr}}) = \left\{ B \in Mat(n, m, R/J) \mid BA = \mathbb{O}_{n \times n}, AB = \mathbb{O}_{m \times m} \right\} = \{0\}$$

$$G = \mathcal{G}_{lr} : \quad Ker^{(l)}(\mathcal{A}_{\mathcal{G}_{lr}}) = \left\{ B \in Mat(n, m, R/J) \mid BA = \mathbb{O}_{n \times n}, AB = \mathbb{O}_{m \times m}, trace(B\vec{\partial}A) = \vec{0} \right\} = \{0\}$$

(Here $\vec{0}$ is the vector of zeros, $\vec{\partial}A$ denotes the vector of all the possible derivations of A . All the matrices/vectors have their entries in R/J .)

Proof. This is immediate consequence of lemma 2.9. We should only check that writing the left kernel of $\mathcal{A}_{G_{lr}}$, $\mathcal{A}_{\mathcal{G}_{lr}}$ in the matrix form,

$$(w_1, \dots, w_{mn}) \rightarrow \begin{pmatrix} w_1 & \dots & w_n \\ w_{n+1} & \dots & w_{2n} \\ \dots & \dots & \dots \\ w_{n(m-1)+1} & \dots & w_{mn} \end{pmatrix},$$

gives the prescribed equations. ■

3. IMPLICATIONS FOR PARTICULAR GROUPS AND SUBSPACES OF $Mat(m, n, R)$

3.1. The case of G_{lr} .

Theorem 3.1. 1. *If $m < n$ then no $A \in Mat(m, n, \mathfrak{m})$ is finitely G_l determined.*

2. *$A \in Mat(m, n, \mathfrak{m})$ is finitely- G_{lr} -determined iff it is finitely- G_r -determined.*

3. *$A \in Mat(m, n, \mathfrak{m})$ is finitely- G_r -determined iff $I_m(A)$ contains a power of the maximal ideal of R .*

Note that in the last statement the ideal of maximal minors is taken for A , not for \mathcal{A} .

Proof. By lemma 2.5 we need to compute the ideal of maximal minors of \mathcal{A} . In the absence of derivations (of $Aut(R)$), $\mathcal{A}_{G_{lr}}$ is a $mn \times (m^2 + n^2)$ matrix, (and $mn < m^2 + n^2$), so a minor is specified by the choice of mn columns of $\mathcal{A}_{G_{lr}}$.

1. Note that the G_l part is a $mn \times m^2$ matrix, thus, if $m < n$ we get $I_{mn}(\mathcal{A}_{G_l}) = 0$. In particular, (as the ring R is of positive dimension, i.e. not Artinian), $I_{mn}(\mathcal{A}_{G_l})$ can never contain a power of \mathfrak{m} .

2. The direction \Leftarrow is trivial, as $G_{lr} \supset G_r$. For the direction \Rightarrow , suppose A is not finitely G_r -determined, then there exists a radical ideal $J \subset R$ with a non-zero left kernel vector, $0 \neq v \in \text{Ker}^{(l)}(A \otimes R/J)$. As $n \geq m$, the ordinary kernel is non-zero too, i.e. there exists $0 \neq u \in \text{Ker}(A \otimes R/J)$ which is supported on $V(J)$. Namely, if $gu = 0 \in (R/J)^{\oplus n}$ then $g = 0 \in R/J$ (recall that J is a radical ideal). Therefore, the matrix $B = u \otimes v = (\{u_i v_j\}_{ij}) \in \text{Mat}(n, m, R/J)$ is non-zero and satisfies: $AB = 0$, $BA = 0$. Thus, A is not finitely- G_{lr} -determined.

3. Using the second form of \mathcal{A} as above, we get: $I_{\max}(\mathcal{A}_{G_r}) = (I_m(A))^n$. In particular, $I_m(A)$ contains a power of the maximal ideal iff $I_{\max}(\mathcal{A}_{G_r})$ does. Hence A is G_r finitely determined iff the ideal $I_{\max}(A)$ contains a power of $\mathfrak{m} \subset R$. ■

Example 3.2. Consider the trivial case: A is a constant (numeric) matrix, $m \leq n$. Then A is finitely G_{lr} -determined iff at least one of its maximal minors is a non-zero constant, i.e. A is of the full rank. (In other words, A is invertible from the left, i.e. it has no left-kernel.) In this case, for $m \leq n$, A is 0-determined with respect to G_r , i.e. stable.

Example 3.3. Another trivial case is $\dim(R) = 1$. In this case if $f \in R$ is not a zero divisor then $fR \supset \mathfrak{m}^N$ for some $N > 0$. (This follows from the existence of conductor in R .) So, a matrix is finitely G_r -determined if at least one of its maximal minors is not a zero divisor in R .

More generally we have:

Corollary 3.4. Suppose $\dim(R) > 0$.

1. If $m = n$ then $A \in \text{Mat}(m, m, \mathfrak{m})$ is finitely G_{lr} -determined iff $\dim(R) = 1$ and $\det(A) \in R$ is not a zero divisor.
2. If $\dim(R) > |n - m| + 1$ then no matrix in $\text{Mat}(m, m, \mathfrak{m})$ is finitely G_{lr} determined.
3. Suppose $\dim(R) \leq |n - m| + 1$. Given $A \in \text{Mat}(m, n, R)$ and $N > 0$, for any generic enough $B \in \text{Mat}(m, n, \mathfrak{m}^N)$, the matrix $A + B$ is finitely G_r -determined. In particular, the set of matrices that are not finitely determined is of infinite codimension in $\text{Mat}(m, n, R)$.
4. Suppose $\dim(R) = 2$. If A has at least two maximal minors whose determinants are relatively prime, (i.e. if $\Delta_i = a_i h \in R$ then $h \in R$ is invertible), then A is finitely G_r -determined.

Proof. (1) is immediate.

(2). Assume $m \leq n$. If $I_m \supset \mathfrak{m}^N$ for some $N > 0$ then the germ defined by this ideal, $V(I_m) \subset \text{Spec}(R)$, is supported at the origin only, i.e. the dimension of $V(I_m)$ is zero. But the codimension of the germ defined by the maximal minors is at most $(n - m + 1)$. So $\dim V(I_m) \geq \dim(R) - (n - m + 1)$, i.e. $\dim(R) - (n - m + 1) \leq 0$.

(3). Follows by observation that for generic enough B the codimension of $I_m(A + B)$, or of each irreducible component of this variety, is the expected one.

(4). Let Δ_1, Δ_2 be two such minors then the scheme $\{\Delta_1 = 0 = \Delta_2\}$ is supported at the origin only, $0 \in \text{Spec}(R)$. Thus the local ring contains a power of maximal ideal. ■

3.2. Finite determinacy for deformations that preserve $I_m(A)$. Recall that G_{lr} preserves all the fitting ideals, (in particular, in most cases there are no finitely- G_{lr} -determined matrices). Therefore, for the action of G_{lr} or of its subgroups, it is natural to consider deformations of $A \in \text{Mat}(m, n, \mathfrak{m})$ only inside the stratum

$$(5) \quad \Sigma_{I_m(A)} := \{B \in \text{Mat}(m, n, \mathfrak{m}) \mid I_m(B) = I_m(A)\}.$$

In terms of commutative algebra, we consider only the modules with the given support. Note that now $\Sigma_{I_m(A)}$ is not a linear subspace of $\text{Mat}(m, n, \mathfrak{m})$. Still, by the general criteria, it is enough to check that the embedding of tangent spaces, $T_{(G_{lr}A, A)} \subset T_{(\Sigma_{I_m(A)}, A)}$, is of finite codimension over \mathbb{K} .

Lemma 3.5. $T_{(\Sigma_{I_m(A)}, A)} = \{B \in \text{Mat}(m, n, \mathfrak{m}) \mid \sum_{\blacksquare \subset \{1, \dots, n\}} \text{trace}(A_{\blacksquare}^{\vee} B_{\blacksquare}) \in I_m(A)\}.$

here \blacksquare denotes some $m \times m$ block of the matrix, the sum goes over all such blocks.

Proof. Follows by expansion in powers of ϵ : $I_m(A + \epsilon B) = I_m(A) + \epsilon \sum_{\blacksquare \subset \{1, \dots, n\}} \text{trace}(A_{\blacksquare}^{\vee} B_{\blacksquare}) + \dots$ ■

Proposition 3.6. Let R be a regular ring, for a given $A \in \text{Mat}(m, n, \mathfrak{m})$ consider only deformations inside $\Sigma_{I_m(A)}$. If $\dim(R) > 2(|n - m| + 2)$ then A is never finitely- G_{lr} -determined.

Proof. In this case the ideal $I_{m-1}(A)$ defines a subspace of $\text{Spec}(R)$ of positive dimension, i.e. the Krull dimension of $R/I_{m-1}(A)$ is positive. On the other hand, restricting to $R/I_{m-1}(A)$ we get $T_{(\Sigma_{I_m(A)}, A)} = \text{Mat}(m, n, \mathfrak{m}/I_{m-1}(A))$. Thus, by the general lemma there cannot be G_{lr} finite determinacy. ■

By similar consideration we get: if $\dim(R) > mn$ then there are no finitely- G_{lr} -determined matrices, even if one restricts to deformations that preserve all the fitting ideals.

3.3. Congruence and (anti)symmetric matrices. For square matrices consider the congruence, $A \stackrel{G_{\text{congr}}}{\sim} UAU^T$, and the corresponding group $\mathcal{G}_{\text{congr}} := G_{\text{congr}} \rtimes \text{Aut}(R)$. For $A \in \text{Mat}^{\text{sym}}(n, n, R)$ the matrix $\mathcal{A}_{\mathcal{G}_{\text{congr}}}$ is of size $\binom{m+1}{2} \times m^2$:

$$(6) \quad \mathcal{A}_{\mathcal{G}_{\text{congr}}} = \begin{pmatrix} \vec{\partial}a_{11} \\ \vec{\partial}a_{12} \\ \dots \\ \vec{\partial}a_{1m} \\ \vec{\partial}a_{22} \\ \vec{\partial}a_{23} \\ \dots \\ \vec{\partial}a_{2m} \\ \vec{\partial}a_{33} \\ \dots \\ \vec{\partial}a_{3m} \\ \dots \end{pmatrix} \begin{pmatrix} 2a_{11} & 2a_{12} & \dots & 2a_{1m} & 0 & \dots & \dots & 0 & 0 & \dots & 0 \\ a_{12} & a_{22} & \dots & a_{2m} & a_{11} & a_{12} & \dots & a_{1m} & 0 & \dots & 0 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & 0 & \dots & \dots & 0 & a_{11} & \dots & a_{1m} & \dots & \dots & \dots \\ a_{1m} & a_{2m} & \dots & a_{mm} & 0 & \dots & \dots & 0 & 0 & \dots & 0 & \dots & \dots & \dots \\ & & & & 2a_{12} & 2a_{22} & \dots & 2a_{2m} & 0 & \dots & 0 & & & \\ & & & & a_{13} & a_{23} & \dots & a_{3m} & a_{21} & \dots & a_{2m} & & & \\ & & & & \dots & \dots & \dots & & 0 & \dots & 0 & & & \\ & & & & a_{1m} & a_{2m} & \dots & a_{mm} & 0 & \dots & 0 & & & \\ & & & & & & & & 2a_{31} & \dots & 2a_{3m} & & & \\ & & & & & & & & \dots & \dots & \dots & & & \\ & & & & & & & & a_{m1} & \dots & a_{mm} & & & \\ & & & & & & & & & \bigcirc & & & \dots \end{pmatrix}$$

For $A \in \text{Mat}^{\text{anti-sym}}(n, n, R)$ the matrix $\mathcal{A}_{\mathcal{G}_{\text{congr}}}$ is of size $\binom{m}{2} \times m^2$:

$$(7) \quad \mathcal{A}_{\mathcal{G}_{\text{congr}}} = \begin{pmatrix} \vec{\partial}a_{12} \\ \vec{\partial}a_{13} \\ \dots \\ \vec{\partial}a_{1m} \\ \vec{\partial}a_{23} \\ \vec{\partial}a_{24} \\ \dots \\ \vec{\partial}a_{2m} \\ \vec{\partial}a_{34} \\ \dots \\ \vec{\partial}a_{3m} \\ \dots \end{pmatrix} \begin{pmatrix} a_{12} & a_{22} & \dots & a_{m2} & a_{11} & a_{12} & \dots & a_{1m} & 0 & \dots & 0 \\ a_{13} & a_{23} & \dots & a_{m3} & 0 & \dots & \dots & 0 & a_{11} & \dots & a_{1m} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & 0 & \dots & \dots & 0 & 0 & \dots & 0 & \dots & \dots & \dots \\ a_{1m} & a_{2m} & \dots & a_{mm} & 0 & \dots & \dots & 0 & 0 & \dots & 0 & \dots & \dots & \dots \\ & & & & a_{13} & a_{23} & \dots & a_{m3} & a_{21} & \dots & a_{2m} & & & \\ & & & & a_{14} & a_{24} & \dots & a_{m4} & 0 & \dots & 0 & & & \\ & & & & \dots & \dots & \dots & & 0 & \dots & 0 & & & \\ & & & & a_{1m} & a_{2m} & \dots & a_{mm} & 0 & \dots & 0 & & & \\ & & & & & & & & a_{14} & \dots & a_{m4} & & & \\ & & & & & & & & \dots & \dots & \dots & & & \\ & & & & & & & & a_{1m} & \dots & a_{mm} & & & \\ & & & & & & & & & \bigcirc & & & \dots \end{pmatrix}$$

Proposition 3.7. 1. For $\dim(R) < mn$ there are no finitely- $\mathcal{G}_{\text{congr}}$ -determined matrices in $\text{Mat}(m, n, \mathfrak{m})$.

2. A symmetric matrix A is finitely $\mathcal{G}_{\text{congr}}$ -determined (in $\text{Mat}^{\text{sym}}(m, m, R)$) iff for any radical ideal $J \subset R$ that is not a maximal ideal the following kernel module is trivial:

$$\left\{ B \in \text{Mat}^{\text{sym}}(m, m, R/J) \mid BA = \mathbb{O}_{m \times m}, \text{trace}(B\vec{\partial}A) = 0 \right\} = \{0\}.$$

3. Similarly, an anti-symmetric matrix A is finitely $\mathcal{G}_{\text{congr}}$ -determined (in $\text{Mat}^{\text{anti-sym}}(m, m, R)$) iff

$$\left\{ B \in \text{Mat}^{\text{anti-sym}}(m, m, R/J) \mid BA = \mathbb{O}_{m \times m}, \text{trace}(B\vec{\partial}A) = 0 \right\} = \{0\}.$$

Proof. 1. Consider the decomposition into (anti)symmetric parts: $A = A_+ + A_-$. This decomposition is preserved by $\mathcal{G}_{\text{congr}}$, i.e. $A \rightarrow UA_+(\phi(x))U^T + UA_-(\phi(x))U^T$, with $UA_+(\phi(x))U^T \in \text{Mat}^{\text{sym}}(m, m, \mathfrak{m})$ and $UA_-(\phi(x))U^T \in \text{Mat}^{\text{anti-sym}}(m, m, \mathfrak{m})$. Therefore, if A is finitely $\mathcal{G}_{\text{congr}}$ -determined (in $\text{Mat}(m, n, R)$) then the pair (A_+, A_-) is finitely determined too. Suppose A is generic enough, consider the pair of the hypersurface germs $V_{\pm} := \{\det(A_{\pm}) = 0\} \subset \text{Spec}(R)$. Finite determinacy of A would imply the finite determinacy of this pair, i.e. for high-order-terms-deformations $V_{\pm}(\epsilon)$, induced by deformations of A_{\pm} , there exists *one* change of variables $\phi \in \text{Aut}(R)$ satisfying $\phi(V_{\pm}(\epsilon)) = V_{\pm}$. For example, for $V_+(\epsilon) = V_+$, the automorphism ϕ should both preserve V_+ and adjust V_- . Which is impossible for generic choices.

2. and 3. follow immediately from the form of $\mathcal{A}_{\mathcal{G}_{\text{congr}}}$ as above. ■

Proposition 3.8. 1. For $A \in \text{Mat}^{\text{sym}}(m, m, R)$: $I_m(A) \supsetneq I_{\max}(\mathcal{A}_{\mathcal{G}_{\text{congr}}}) \supseteq I_m(A)I_{m-1}(A) \cdots I_1(A)$.

2. In particular, $A \in \text{Mat}^{\text{sym}}(m, m, R)$ is finitely- $\mathcal{G}_{\text{congr}}$ -determined iff $I_m(A)$ contains a power of the maximal ideal.

3. For $A \in \text{Mat}^{\text{anti-sym}}(m, m, R)$: $I_{m-1}(A) \supset I_{\max}(\mathcal{A}_{\mathcal{G}_{\text{congr}}}) \supseteq I_{m-1}(A)I_{m-2}(A) \cdots I_1(A)$.

4. In particular, $A \in \text{Mat}^{\text{anti-sym}}(m, m, R)$ is finitely- $\mathcal{G}_{\text{congr}}$ -determined iff $I_{m-1}(A)$ contains a power of the maximal ideal.

Proof. 1. The matrix $\mathcal{A}_{\mathcal{G}_{\text{congr}}}$ is of size $\binom{m+1}{2} \times m^2$, therefore any of its maximal minor is a choice of $\binom{m+1}{2}$ columns. Note that $\mathcal{A}_{\mathcal{G}_{\text{congr}}}$ is naturally split into $\binom{m+1}{2} \times m$ blocks of columns. Choose some (arbitrary) ordering of these blocks, $\{\mathcal{A}^{(j)}\}_{j=1 \dots m}$. One of the ways to choose a set of columns is by the distribution, $\binom{m+1}{2} = m + (m-1) + \dots + 1$, taking j columns from j 'th block of columns. To compute the determinant of such a minor, start from the block $\mathcal{A}^{(m)}$, its

contribution is $\det(\mathcal{A}^{(m)})$. Note that the blocks $\mathcal{A}^{(m)}$ and $\mathcal{A}^{(m-1)}$ have precisely one common row (which is "taken" when computing $\det(\mathcal{A}^{(m)})$), thus the contribution of $\mathcal{A}^{(m-1)}$ is the determinant of the remaining $(m-1) \times (m-1)$ block. Continue in this way up to $\mathcal{A}^{(1)}$, multiplying all the contributions we get an element of $I_m(A)I_{m-1}(A) \dots I_1(A)$. By going over all the distributions of columns (for the given ordering $\{\mathcal{A}^{(j)}\}_{j=1 \dots m}$) and then over all the orderings of $\{\mathcal{A}^{(j)}\}_{j=1 \dots m}$ we can realize all the elements of $I_m(A)I_{m-1}(A) \dots I_1(A)$. Thus $I_{\max}(\mathcal{A}_{G_{\text{congr}}}) \supseteq I_m(A)I_{m-1}(A) \dots I_1(A)$.

To show that $I_m(A) \supset I_{\max}(\mathcal{A}_{G_{\text{congr}}})$ it is enough to notice that for generic A the ideal $I_m(A)$ is radical and to check that $\mathcal{A}_{G_{\text{congr}}} \otimes R/I_m(A)$ has non-trivial left kernel. (Recall that the support of the left kernel is precisely the ideal of maximal minors.) By proposition 3.7 the left kernel in this case consists of symmetric matrices satisfying $BA = \mathbb{O} \in \text{Mat}(m, m, R/I_m(A))$. The natural candidate is the adjugate matrix, $A^\vee A = \det(A)\mathbb{I} = AA^\vee$. As A is symmetric, A^\vee is symmetric too, hence it provides the left kernel, except in the case $A^\vee = \mathbb{O} \in \text{Mat}(m, m, R/I_m(A))$. But then, immediately, $\det(A) = 0 \in R$. So, for any $J \subsetneq \mathfrak{m} \subset R$ the matrix $A \otimes R/J$ cannot be finitely determined. Which means, by lemmas 2.5 and 2.9, that the image of $I_{\max}(\mathcal{A}_{G_{\text{congr}}})$ in R/J does not contain a power of maximal ideal. Which means $I_{\max}(\mathcal{A}_{G_{\text{congr}}}) = \{0\} \subset R$. Thus $I_m(A) \supset I_{\max}(\mathcal{A}_{G_{\text{congr}}})$.

2. Follows immediately from the first part and lemma 2.5.

3. The proof of $I_{\max}(\mathcal{A}_{G_{\text{congr}}}) \supseteq I_{m-1}(A)I_{m-2}(A) \dots I_1(A)$ goes as for part 1. So, we only need to prove $I_{m-1}(A) \supset I_{\max}(\mathcal{A}_{G_{\text{congr}}})$.

Suppose m is even, then for generic antisymmetric matrix $\det(A) \neq 0 \in R$, while A^\vee is antisymmetric too. Therefore $A \otimes R/\sqrt{I_m(A)}$ has the kernel: A^\vee . As $\sqrt{I_m(A)} \subset \sqrt{I_{m-1}(A)}$ we get: $A \otimes R/\sqrt{I_{m-1}(A)}$ has non-trivial kernel too, thus is not finitely- G_{congr} -determined.

If m is odd, then $\det(A) = 0 \in R$ for any antisymmetric matrix (and A^\vee is symmetric), thus we should construct the kernel in a different way. Let $A^{(i,i)}$ be the $(m-1) \times (m-1)$ block of A obtained by erasing i 'th row and column. It is an antisymmetric matrix and its adjugate, $(A^{(i,i)})^\vee$, is antisymmetric too. Complete $(A^{(i,i)})^\vee$ to the $m \times m$ matrix, by inserting at i 'th position the zero row and column, denote the resulting matrix by B . By construction, B is an antisymmetric matrix and over $R/\sqrt{I_{m-1}(A)}$ the columns $1, \dots, \hat{i}, \dots, m$ of the product BA are zero. But $\det(A) = 0 \in R$, i.e. its columns are linearly dependent. So, if B kills (over $R/\sqrt{I_{m-1}(A)}$) all the columns except for i 'th, then it kills the i 'th column too. Finally, as we assume $I_{m-1}(A) \neq \{0\}$, the antisymmetric matrix B is not zero. Providing the needed left kernel of A .

4. Follows now immediately from the third part and lemma 2.5. ■

3.4. Upper triangular matrices. Let $\text{Mat}^{up}(m, m, R) \subset \text{Mat}(m, m, R)$ be the R -submodule of upper triangular matrices, $A_{ij} = 0$ for $i > j$, let $G_{lr}^{up} \subset G_{lr}$ be the group of transformations $A \rightarrow UAV$ with U, V -upper triangular. Identify $\text{Mat}^{up}(m, m, R)$ with $R^{\binom{m+1}{2}}$ by choosing the basis $(a_{11}, a_{12}, \dots, a_{1m}, a_{22}, \dots, a_{2m}, \dots, a_{mm})$. In this basis the generating matrix of $T_{(G_r^{up} A, A)}$ and $T_{(G_l^{up} A, A)}$ are:

$$(8) \quad \mathcal{A}_{G_r^{up}} = \begin{pmatrix} a_{11} & 0 & 0 & & & & \\ 0 & a_{11} & a_{12} & & \mathbb{O} & \mathbb{O} & \dots \\ 0 & 0 & a_{22} & & & & \\ & \mathbb{O} & & a_{11} & a_{12} & a_{13} & \\ & & & 0 & a_{22} & a_{23} & \mathbb{O} \\ & & & 0 & 0 & a_{33} & \\ \dots & & & \dots & \dots & \dots & \dots \\ & \mathbb{O} & & \mathbb{O} & & \dots & a_{11} \ a_{12} \ \dots \ a_{1m} \\ & & & & & 0 & a_{22} \ \dots \ a_{2m} \\ & & & & & \dots & \dots \ \dots \\ & & & & & 0 & \dots \ 0 \ a_{mm} \end{pmatrix}$$

$$(9) \quad \mathcal{A}_{G_l^{up}} = \begin{pmatrix} a_{11} & 0 & 0 & & & & \\ a_{12} & a_{22} & 0 & & \mathbb{O} & \mathbb{O} & \dots \\ \dots & \dots & \dots & & & & \\ a_{1m} & a_{2m} & \dots & a_{mm} & & & \\ & & & & a_{22} & 0 & 0 \\ & & & & a_{23} & a_{33} & 0 \\ & & & & \dots & \dots & \dots \\ & & & & a_{2m} & a_{3m} & \dots \ a_{mm} \\ & & & & & & & \mathbb{O} \\ & & & & & & & \dots \ \mathbb{O} \\ & & & & & & & \dots \ a_{mm} \end{pmatrix}$$

Thus we get immediate:

Corollary 3.9. 1. $I_{\max}(\mathcal{A}_{G_r^{up}}) = \prod_{j=1}^m (\prod_{i=1}^j a_{ii})$ and $I_{\max}(\mathcal{A}_{G_l^{up}}) = \prod_{j=1}^m (\prod_{i=j}^m a_{ii})$.

2. In particular, A is G_r^{up} or G_l^{up} finitely determined, inside $\text{Mat}^{up}(m, m, R)$, iff $I_m(A)$ contains a power of the maximal

ideal of R .

3. A is finitely G_{lr}^{up} determined iff for any radical ideal $J \subset R$ the kernel module is trivial

$$\{B \in Mat^{up}(m, m, R/J) \mid l.t.(B^T A) = 0, u.t.(BA^T) = 0\} = \{0\},$$

here $l.t.$ and $u.t.$ are the lower triangular and upper triangular parts.

3.5. The case of $Aut(R)$. In this case the group does not use any matrix structure, we have just the classical right equivalence, $Aut(R) \circ Maps(Spec(R), (\mathbb{K}^n, 0))$. For completeness we reprove the corresponding statements.

Let $\{x_i\}_{i=1..k}$ be a sequence of "regular parameters" in R , i.e. a sequence of elements such that the quotient $R/(x_1, \dots, x_k)$ is of Krull dimension $(\dim(R) - k)$ and the images of the generators $\{x_i\}_i$ in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent. (If R is regular then $\{x_i\}$ is just any system of generators, in general this is a system of coordinates on the Noether normalization of a maximal dimension component of $Spec(R)$.)

Proposition 3.10. 1. If $n > 1$ then the map $A \in (Spec(R), (\mathbb{K}^n, 0))$ is finitely- $Aut(R)$ -determined iff $\dim(R) \geq n$ and the entries of A form a sequence of regular parameters (of length n) in the sense above.

2. Moreover, in the later case A is $Aut(R)$ -stable.

3. If $n = 1$ then the map $A \in (Spec(R), (\mathbb{K}^1, 0))$, i.e. an element of the ring, is finitely- $Aut(R)$ -determined iff the ideal $(\partial_1 A, \dots, \partial_p A)$ contains a power of \mathfrak{m} , i.e. defines a one-point-scheme on $Spec(R)$.

(If R is not regular then $A^{-1}(0)$ is not necessarily reduced or has an isolated singularity, cf. example 3.15.)

Proof. 1. By lemma 2.5 we should check the ideal of maximal minors, $I_{max}(\mathcal{A}_{Aut(R)})$, where $\mathcal{A}_{Aut(R)}$ is a $n \times p$ matrix. In particular, if $n > p$, then $I_{max}(\mathcal{A}_{Aut(R)}) = \{0\}$, i.e. A is not finitely- $Aut(R)$ -determined. For $n \leq p$ this ideal defines the subgerm of $Spec(R)$, which is either empty or of codimension at most $(p - n + 1)$.

The first case means that $I_{max}(\mathcal{A}_{Aut(R)}) = R$, i.e. the determinant of one of the minors of $\mathcal{A}_{Aut(R)}$ is an invertible element of R . As the elements are derivatives, we get that the entries of A form a sequence of regular parameters.

In the second case the ideal $I_{max}(\mathcal{A}_{Aut(R)})$ defines a subgerm of positive dimension (as $n > 1$), thus the ideal cannot contain any power of $\mathfrak{m} \subset R$.

2. As we have the system of regular parameters $\{x_i\}$, any change $\{x_i \rightarrow x_i + \phi_i\}$, with $\phi_i \in \mathfrak{m}$, lifts to an automorphism of R . Hence for any $B \in Maps(Spec(R), (\mathbb{K}^n, 0))$ there exists $\phi \in Aut(R)$ such that $\phi(A + B) = A$. ■

3.6. The case of \mathcal{G}_r .

Corollary 3.11. 1. If $p < n$ then A is finitely- \mathcal{G}_r -determined iff it is finitely G_r -determined. In particular, for $n - m + 1 < \dim(R) < n$ there are no finitely- \mathcal{G}_r -determined matrices.

2. More generally, if $p < kn$ and A is finitely- \mathcal{G}_r -determined then the ideal $I_{m-k+1}(A)$ contains a power of the maximal ideal of R . Thus $\dim(R) \leq k(n - m + k)$.

Proof. 1. If $p < mn$ then any $mn \times mn$ minor of $\mathcal{A}_{\mathcal{G}_r}$ contains at least $mn - p + 1$ columns of \mathcal{A}_{G_r} , so for $p < n$ it contains at least one block of A . So, the minor belongs to $I_m(A)$. Thus $I_{max}(\mathcal{A}_{\mathcal{G}_r}) \subset I_m(A)$. On the other hand, $(I_m(A))^n \subset I_{max}(\mathcal{A}_{\mathcal{G}_r})$, thus $I_{max}(\mathcal{A}_{\mathcal{G}_r})$ contains a power of \mathfrak{m} iff $I_m(A)$ does.

The second statement follows from corollary 3.4.

2. In the same way, check the minors of \mathcal{A} to get $I_{max}(\mathcal{A}) \subset I_{m-k+1}(A)$. ■

3.7. The case of \mathcal{G}_{lr} .

3.7.1. Conditions on fitting ideals. If $A \in Mat(m, n, R)$ is \mathcal{G}_{lr} -finitely determined then all the fitting ideals are "simultaneously finitely determined", i.e. for any $B_{\geq N}$ there exists $\phi \in Aut(R)$ satisfying:

$$(10) \quad \{\phi(I_1(A + B_{\geq N})), \dots, \phi(I_m(A + B_{\geq N}))\} = \{\phi(I_1(A)), \dots, \phi(I_m(A))\}$$

On this occasion we record the miniversal deformations of these ideals.

Lemma 3.12. The tangent space of the miniversal deformation of $I_j(A)$ is $I_{j-1}(A) / \langle I_j(A), \vec{\partial} I_j(A) \rangle$.

Proof. Consider the deformation $A + \epsilon B$, expand in powers of ϵ and use (for each square block) the relation $\det(A + \epsilon B) = \det(A) + \epsilon \text{tr}(A^\vee B) + \dots$. By choosing all the possible matrices B we get: $T_{I_j(A)}$ is spanned by all the elements of $I_{j-1}(A)$.

The tangent space to the orbit $Aut(R)I_j(A)$ is computed as usual. Consider an infinitesimal automorphism, whose action on the generators of R is: $x_i \rightarrow x_i + \epsilon \phi_i(\underline{x})$. Then $\det(A) \rightarrow \det(A) + \epsilon \sum_i \phi_i(\underline{x}) \partial_i \det(A)$. In addition, the change $(1 + \epsilon u(\underline{x})) \det(A)$ does not change the ideal. Hence the statement. ■

For an ideal $J \subset R$ denote by $\text{ord}(J)$ the maximal number k such that $J \subset \mathfrak{m}^k$.

Corollary 3.13. Suppose R is a regular ring with some fixed generators (x_1, \dots, x_p) . Suppose $\text{ord}(I_m(A)) > \text{ord}(I_{m-1}(A)) + 1$, and moreover no maximal minor of A involves x_p . Then A is not finitely- \mathcal{G}_{lr} -determined.

Proof. Let $a \in I_{m-1}(A)$ be an element whose order coincides with that of $I_{m-1}(A)$. By lemma 3.12 the miniversal deformation of $I_m(A)$ contains the (image of the) subspace generated by the elements $\{ax_p^j\}_{j \geq 0}$. On the other hand, by initial assumption: if $a(\sum_j c_j x_p^j) \in I_m(A)$, $\vec{\partial} I_m(A) >$ then $a \in I_m(A)$, $\vec{\partial} I_m(A) >$. Which is impossible, as $\text{ord}(a) < \text{ord}(I_m(A)) - 1 = \text{ord}(\vec{\partial} I_m(A))$. Thus the vector subspace (of all the possible deformations) generated by $\{ax_p^j\}_{j \geq 0}$ intersects trivially the subspace $< I_m(A), \vec{\partial} I_m(A) >$. But then the vector space $I_{j-1}(A) \setminus \langle I_j(A), \vec{\partial} I_j(A) \rangle$ is of infinite dimension. ■

Proposition 3.14. *If $A \in \text{Mat}(m, n, R)$ is \mathcal{G}_{lr} -finitely determined then:*

1. *There exists an automorphism $\phi \in \text{Aut}(R)$ such that $\phi(I_1), \dots, \phi(I_m)$ are generated by polynomials.*
2. *The orbit of I_m under $\text{Aut}(R)$ contains a power of maximal ideal: $\text{Aut}(R)I_m \supset \mathfrak{m}^N$*
3. *$\text{codimSing}(V(I_j)) = (m - j + 2)(n - j + 2)$.*
4. *The co-dimensions of all the fitting ideals, $I_j(A)$, are the 'expected' ones. In particular, either $p > mn$ and $I_1(A)$ is a complete intersection ideal or $I_1(A)$ contains a power of maximal ideal.*

Proof. 4. Suppose $I_1(A)$ does not contain any power of \mathfrak{m} . Let $v \in \mathfrak{m}^{N+1}$ such that $v \notin I_1(A)$. Then $I_1(A + \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & v \end{pmatrix}) \neq I_1(A)$. If A is N -determined then there exists $\phi \in \text{Aut}(R)$ such that $\phi(I_1(A)) = I_1(A + \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & v \end{pmatrix})$. In particular, these ideals have the same (co)-dimension. As we can choose v generically, and put it at any position in the matrix, we get: $I_1(A)$ is a complete intersection. ■

Example 3.15. If R is not a regular ring then being finitely determined does not imply that the ideal $I_1(A)$ is radical (i.e. the zero locus $A^{-1}(0)$ is reduced). For example, let $R = \mathbb{K}[[x, y, z]]/(xz, yz)$ and $A \in \text{Mat}(1, 1, R)$ be just $x + y + z$. Then A is obviously finitely-determined (even stable), but $I_1(A)$ defines a non-reduced scheme, whose local ring is $\mathbb{K}[[x, y, z]]/(xz, yz, x+y+z) \approx \mathbb{K}[[x, y]]/(x^2, xy)$. The natural guess is that if R is Cohen-Macaulay then finite-determinacy implies that $I_1(A)$ defines a reduced scheme (or a fat point).

3.7.2. *Finite determinacy with respect to \mathcal{G}_{lr} .*

Proposition 3.16. *If $\dim(R) \leq |n - m| + 1$, then A is finitely \mathcal{G}_{lr} determined iff it is finitely \mathcal{G}_r determined.*

Proof. By theorem 3.1 if A is not finitely \mathcal{G}_r determined then $I_m(A)$ contains no power of the maximal ideal. But for any N , for generic enough $B \in \text{Mat}(m, n, \mathfrak{m}^{N+1})$, the ideal $I_m(A + B)$ defines a one-point scheme, i.e. contains a power of maximal ideal, hence is not equivalent to $I_m(A)$. ■

Proposition 3.17. 1. *Suppose $\dim(R) = n - m + 2 \geq 2$, $m \leq n$, and the ideal $I_m(A)$ defines a one-dimensional generically reduced subspace of $\text{Spec}(R)$ (i.e. the singularity is isolated). Then A is finitely \mathcal{G}_{lr} determined.*
 2. *In particular, for $\dim(R) = n - m + 2$, the set of non-finitely- \mathcal{G}_{lr} -determined matrices is of infinite codimension in $\text{Mat}(m, n, R)$.*

Note that in the second statement we need the whole \mathcal{G}_{lr} , not just \mathcal{G}_r .

Proof. As the germ $V(I_m(A))$ is of expected dimension, $1 = \dim(R) - (n - m + 1)$, any deformation of the matrix, $A \rightarrow A + B$, induces a flat deformation of the ideal, $I_m(A) \rightarrow I_m(A + B)$. This follows from the classical fact: if $V(I_m(A))$ is of expected codimension, i.e. $I_m(A)$ has a regular sequence of expected length then the Eagon-Northcot complex is exact, [Eisenbud-book, §A2].

As the singularity of $V(I_m(A))$ is isolated, this germ is finitely determined with respect to $\text{Aut}(R)$, i.e. has a finite dimensional miniversal deformation. Thus for B of high enough order: $I_m(A) \xrightarrow{\text{Aut}(R)} I_m(A + B)$. By §1.3.2 this germ finitely- $\text{Aut}(R)^{(k)}$ -determined: for any $k > 0$ there exists $N \gg k$, such that if $B \in \text{Mat}(m, n, \mathfrak{m}^{N+1})$ then $\exists \phi \in \text{Aut}(R)$ with $I_m(A) = \phi^* I_m(A + B)$ and $\pi_k(\phi) = \text{Id}$. (Meanwhile keep k indefinite, later we choose it big enough.) Thus $\phi^*(A + B) = A + C_{\geq k}$, and it is enough to prove:

$$(11) \quad \text{if } I_m(A) = I_m(A + C_{\geq k}) \text{ then } A \stackrel{\mathcal{G}_r}{\sim} A + C_{\geq k}$$

Restrict onto $R/I_m(A)$, as mentioned in the introduction, it is enough to prove the \mathcal{G}_{lr} equivalence of restrictions of A and $A + C_{\geq k}$. Note that now all the maximal minors of A (and of $A + C_{\geq k}$) vanish. However, as the ring $R/I_m(A)$ is generically reduced, i.e. the singularity is isolated, there exists at least one $(m - 1) \times (m - 1)$ minor in A , whose determinant is not a zero divisor in $R/I_m(A)$. Using this minor we kill all the entries of $C_{\geq k}$ as follows.

By permutations of rows and columns we can assume that this minor, $A_{(m-1), (m-1)}$, is composed of the first $(m - 1)$ columns and rows of A . As the ring $R/I_m(A)$ is one-dimensional it has the property of conductor: for any non-zero divisor $f \in R/I_m(A)$ there exists $k > 0$ such that $\mathfrak{m}^k \subset (f)$, i.e. any element of this power of maximal ideal is divisible by

f . Thus, for $f = \det(A_{m-1, m-1})$ and $A + C_{\geq k}$ we choose k satisfying this property, so $C_{\geq k}$ is divisible by $\det(A_{m-1, m-1})$. Let $A_{m-1, m-1}^\vee$ be the adjoint matrix of this minor, consider the G_r transformation:

$$(12) \quad A \rightarrow A \left(\mathbb{I}_{n \times n} + \begin{pmatrix} A_{m-1, m-1}^\vee & \frac{C_{m-1, n}}{\det(A_{m-1, m-1})} \\ \mathbb{O} & \end{pmatrix} \right)$$

where the last term is a $n \times n$ matrix, whose first $(m-1)$ rows are of the prescribed type, while all the rest are zeros. The result of this transformation is $A + C'$ where the first $(m-1)$ rows of C and C' coincide and only the last row is possibly different. Now, by a similar G_l transformation we achieve the form $A + C'$, where C' possibly differs from C only at the last $(n-m)$ elements of the last row. So, we have proved: $A + C$ is G_{lr} equivalent to $A + B$, where $B_{ij} = 0$ for $i < m$ or $j < m$.

As we work over $R/I_m(A)$, we have: $I_m(A + C) = I_m(A + B) = I_m(A) = \{0\}$, i.e. all the maximal minors of $A + B$ have zero determinants. In particular, by considering all the minors that include the first $(m-1)$ rows, we get $B_{m, j} \det(A_{(m-1) \times (m-1)}) = 0$ for any $j \geq m$. As $\det(A_{(m-1) \times (m-1)})$ is not a zero divisor, we get: $B = \mathbb{O}$. ■

Corollary 3.18. *Suppose $\dim(R) \leq 4$. Then $A \in \text{Mat}(m, m, R)$ is finitely G_{lr} -determined iff both the ideal $I_{m-2}(A)$ and the ideal generated by $\{\text{trace}(A^\vee \partial_i A)\}_i$ contain a power of the maximal ideal of R . (Alternatively, both ideals define a subspace of $\text{Spec}(R)$ of dimension zero.)*

Proof. By corollary 2.11 we should check the triviality of the kernel $\{BA = \mathbb{O} = AB, \text{trace}(B\vec{\partial}A) = 0\}$, for any restriction to R/J . The first equations imply in particular that $\det(A) = 0 \in R/J$, thus $J \supset (\det(A))$. Further, if $J \not\supset I_{m-2}(A)$, then the co-rank of $A \otimes R/J$ is one, hence from the first two equations we get: $B = gA^\vee$, for some $g \in R/J$. Then, if ideal $\{\text{trace}(A^\vee \partial_i A)\}_i$ does not contain a power of the maximal ideal in $R/(\det(A))$, we get the non-zero kernel. And vice versa. ■

3.8. Applications to the singularities of maps. Suppose $m = 1$, then $\text{Mat}(1, n, \mathfrak{m})$ can be considered as $\text{Maps}(\text{Spec}(R), (\mathbb{K}^n, 0))$. Note that in this case the G_{lr} and G_r equivalences coincide. Moreover, they coincide with the classical *contact* equivalence \mathcal{K} . Therefore in this case we consider the $1 \times n$ matrices as maps.

Corollary 3.19. *Let $\text{Spec}(R) \xrightarrow{A} (\mathbb{K}^n, 0)$, so $A = (a_1, \dots, a_n)$.*

1. *A is finitely- G_r -determined iff $A^{-1} \subset \text{Spec}(R)$ is a (fat) point. In particular, if $n < \dim(R)$ then no such matrix is finitely- G_r -determined. If $n \geq \dim(R)$ then the generic matrix is finitely determined.*
2. *If $p < n$ then A is finitely- \mathcal{K} -determined iff it is finitely- G_r -determined.*
3. *A is finitely- \mathcal{K} -determined iff the ideal defined by $\{a_i\}$ and the ideal of maximal minors of the Jacobian matrix, $\{\partial_j a_i\}_{\substack{i=1, \dots, p \\ j=1, \dots, n}}$ contains \mathfrak{m}^N , for some N . In particular, the generic matrix is finitely- \mathcal{K} -determined. If A is finitely- \mathcal{K} -determined and $p \geq n$ then $A^{-1}(0)$ is a complete intersection (of codimension n).*

Proof. 1. This is just the G_{lr} -criterion from theorem 3.1.

2. Note that in this case $\mathcal{A} \in \text{Mat}(n, p + n^2, R)$ is:

$$(13) \quad \underbrace{\left(\partial_i a_j \right)}_{\substack{\text{Jacobian} \\ \text{matrix}}} \begin{pmatrix} a_1 & a_2 & \dots & a_n & 0 & \dots & & & & \\ 0 & 0 \dots & 0 & 0 & a_1 & a_2 & \dots & a_n & & \\ \dots & \dots & \dots & \dots & \dots & & & & & \\ 0 & \dots & \dots & \dots & 0 & 0 & a_1 & a_2 & \dots & a_n \end{pmatrix}$$

Thus, if $p < n$, then any $n \times n$ block of \mathcal{A} contains a column of the G_r part. Thus the radicals of $I_{\max}(\mathcal{A})$ and $I_1(A)$ coincide. Thus, \mathcal{K} -determinacy implies G_r -determinacy. The converse statement is trivial as $G_r \subset \mathcal{K}$.

3. The first statement is just the statement of lemma 2.5 for the case $\text{Mat}(1, n, R)$. Suppose $A^{-1}(0)$ is not of expected dimension, then $\{a_i\}$ are algebraically dependent, i.e. $\sum_i b_i a_i = 0 \in R$, for some $\{b_i\}$ in R . Then, by differentiation we get: $\sum_i b_i \vec{\partial} a_i = 0 \in R/(\{a_i\})$. And not all b_i are zero in $R/(\{a_i\})$, as $\{a_i\}$ is not a regular sequence. Thus the jacobian matrix $\{\partial_j a_i\}_{\substack{i=1, \dots, p \\ j=1, \dots, n}}$ cannot be of the full rank over $R/(\{a_i\})$, i.e. its ideal of maximal minors is included into $I_1(A)$.

Thus, the ideal of maximal minors of \mathcal{A} cannot contain any power of \mathfrak{m} , as the support of $A^{-1}(0)$ is of positive dimension. ■

Remark 3.20. If R is a regular ring, then from (3) of the last lemma we get the classical criterion ([Wall-1981]): A is finitely- \mathcal{K} -determined iff $A^{-1}(0)$ is an isolated complete intersection. For non-regular rings A^{-1} can have non-isolated singularity but still be finitely determined. For example, let $R = \mathbb{K}[[x_1, \dots, x_n]]/(x_1^k)$, then the module of derivations is generated by $(x_1 \partial_1, \partial_2, \dots, \partial_n)$. Consider $A \in \text{Mat}(1, 1, R)$ defined by $x_1 + x_2$. Note that $\mathcal{A} = (x_1, 1, 0, \dots, 0, x_1 + x_2)$, thus $I_{\max}(\mathcal{A}) = R$, hence finite- \mathcal{K} -determinacy. (In fact, A is even $\text{Aut}(R)$ -finitely determined.) But the zero locus is $\{x_1 + x_2 = 0\} \approx \text{Spec}(\mathbb{K}[x_1, x_2, \dots, x_{n-1}]/(x_1^k))$, i.e. is a multiple hyperplane.

Remark 3.21. Using this approach we can check the determinacy of maps with respect to some subgroups of \mathcal{K} . Consider $\phi \in \text{Maps}(\text{Spec}(R), (\mathbb{K}^N, 0))$, suppose $N = mn$, with $m \leq n$. Associate to this map the matrix $A_\phi^{(m,n)} \in \text{Mat}(m, n, R)$. The group $\mathcal{G}_{lr}^{(m,n)} \circ \text{Mat}(m, n, R)$ is obviously a subgroup of $\mathcal{K} \circ \text{Maps}(\text{Spec}(R), (\mathbb{K}^N, 0))$. And for $m > 1$ the orbits of $\mathcal{G}_{lr}^{(m,n)}$ are *much smaller* than those of \mathcal{K} . Thus, the finite $\mathcal{G}_{lr}^{(m,n)}$ -determinacy is a much stronger property than the finite \mathcal{K} -determinacy.

Further, there is the partial ordering among $\{\mathcal{G}_{lr}^{(m,n)}\}_{mn=N}$: if $m = am'$ and $n = n'/a$, with $m \leq n$ and $a \geq 1$, then $\mathcal{G}_{lr}^{(m,n)} \hookrightarrow \mathcal{G}_{lr}^{(m',n')}$. And their orbits embed naturally. So, the "smallest" possible subgroup is $\mathcal{G}_{lr}^{(m,n)}$ with the smallest n , for the given N .

4. STABLE TYPES

We prove the general criterion for $G \circ \text{Mat}(m, n, \mathfrak{m})$. Recall that for $G_1 \subset G_2$, the G_1 stability implies those of G_2 . Therefore we consider the two main cases: G_{lr} and \mathcal{G}_{lr} .

Let $A \in \text{Mat}(m, n, \mathfrak{m})$, choose some elements $x_1, \dots, x_p \in \mathfrak{m}$, whose images generate $\mathfrak{m}/\mathfrak{m}^2$. Accordingly, we expand $A = \sum_i x_i A_i + \dots$, where $\{A_i \in \text{Mat}(m, n, \mathbb{K})\}$ and the dots mean the higher order terms. Combine the matrices $\{A_i\}$ into one *numerical* matrix:

$$(14) \quad \mathcal{B} := \begin{pmatrix} A_1 \\ A_2 \\ \dots \\ A_p \end{pmatrix} \in \text{Mat}(pm, n, \mathbb{K})$$

Proposition 4.1. 1. If $A \in \text{Mat}(m, n, \mathfrak{m})$ is \mathcal{G}_{lr} -stable then $\text{rank}(\mathcal{B}) = \min(n, mp)$.

2. If A is G_{lr} -stable then $\text{rank}(\mathcal{B}) = pm$, in particular $pm \leq n$.

3. If $\text{rank}(\mathcal{B}) = pm$ then A is G_r -stable. In particular, for $pm \leq n$, the \mathcal{G}_{lr} -stability implies the of G_r .

Proof. If $A \in \text{Mat}(m, n, \mathfrak{m})$ is stable then its image in $\text{Mat}(m, n, \mathfrak{m}/\mathfrak{m}^2)$ is stable too. Thus to get the necessary conditions it is enough to check the stability of \mathcal{B} . Note that the G_r part of G_{lr} acts on \mathcal{B} from the right, as $GL(n, \mathbb{K})$, while the G_l part acts from the left simultaneously on all the (horizontal) $m \times n$ blocks, (as $GL(m, \mathbb{K})$). Further, the action of $\text{Aut}(R)$ reshuffles these blocks. Therefore, \mathcal{G}_{lr} acts on \mathcal{B} as the subgroup of $GL(pm, \mathbb{K}) \times GL(n, \mathbb{K})$.

1. One can always deform A so that \mathcal{B} deforms to a matrix of the full rank. But, as said above, \mathcal{G}_{lr} preserves the rank of \mathcal{B} . Hence, A cannot be stable unless \mathcal{B} is of full rank.

2. Suppose $\text{rank}(\mathcal{B}) < pm$, it is enough to check the non-stability of the image of A in $\text{Mat}(m, n, \mathfrak{m}/\mathfrak{m}^2)$. If \mathcal{B} is not of (full) rank pm , then there exists some $1 \leq i < j < p$ such that the $2m \times n$ matrix composed of A_i, A_j is not of full rank too, i.e. its rank is less than $2m$. Geometrically this corresponds to the restriction onto the subspace spanned by x_i, x_j . We can deform A_i, A_j generically, thus for G_{lr} -stability they should be of the full rank. Then the $2m \times n$ block can be brought, e.g., to the form

$$(15) \quad \begin{pmatrix} A_i \\ A_j \end{pmatrix} = \begin{pmatrix} \mathbb{I} & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{I} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{I} & \mathbb{O} \\ * & * & * & \mathbb{O} \end{pmatrix}$$

Then, e.g. the scaling of the last row cannot be undone by the remaining transformations.

3. Suppose \mathcal{B} is of (full) rank pm , then $n \geq pm$ and by $GL(n, \mathbb{K})$ transformations (from the right!) we can bring \mathcal{B} to the "diagonal" form:

$$(16) \quad \mathcal{B} \stackrel{GL(n, \mathbb{K})}{\sim} \begin{pmatrix} \mathbb{I}_{m \times m} & \mathbb{O} & \mathbb{O} & \mathbb{O}_{m \times (n-pm)} \\ \mathbb{O} & \mathbb{I}_{m \times m} & \mathbb{O} & \mathbb{O}_{m \times (n-pm)} \\ \dots & \dots & \dots & \mathbb{O}_{m \times (n-pm)} \\ \mathbb{O} & \dots & \mathbb{I}_{m \times m} & \mathbb{O}_{m \times (n-pm)} \end{pmatrix}$$

This is the (unique) normal form for any \mathcal{B} of rank pm , the later condition is open, thus such a matrix is (right) stable. Thus the image of A in $\text{Mat}(m, n, \mathfrak{m}/\mathfrak{m}^2)$ is G_r -stable. Finally, by direct check, we get that A itself is G_r -stable. ■

5. APPENDIX

5.1. The case of C^∞ and analytic categories. Let R be one of $\mathbb{R}\{x_1, \dots, x_p\}$, $C^\infty(\mathbb{R}^p, 0)$, let \hat{R} be its completion. For $A \in \text{Mat}(m, n, R)$, let $\hat{A} \in \text{Mat}(m, n, \hat{R})$ be the corresponding completion. Let $G = \mathcal{G}_{lr} \circ \text{Mat}(m, n, R)$, let $\hat{G} = \hat{\mathcal{G}}_{lr} \circ \text{Mat}(m, n, \hat{R})$ be the corresponding completion.

Theorem 5.1. 1. For $R = \mathbb{R}\{x_1, \dots, x_p\}$, a matrix $A \in \text{Mat}(m, n, R)$ is analytically finitely- \mathcal{G}_{lr} -determined iff it is formally finitely- $\hat{\mathcal{G}}_{lr}$ -determined.

2. For $R = C^\infty(\mathbb{R}^p, 0)$ a matrix $A \in \text{Mat}(m, n, R)$ is C^∞ finitely- \mathcal{G}_{lr} -determined iff its completion, $\hat{A} \in \text{Mat}(m, n, \hat{R})$, is formally finitely- $\hat{\mathcal{G}}_{lr}$ -determined.

Proof. In both cases the direct statements are immediate, we prove only the converse statements.

1. Suppose $A \in \text{Mat}(m, n, R)$ is formally finitely-determined, then for any $B_{>N}$ there exists a *formal* triple $(\hat{U}, \hat{V}, \hat{\phi}) \in \hat{G} \subset \hat{\mathcal{G}}_{lr}$ satisfying: $\hat{U}A(\hat{\phi}(x))\hat{V} = A(x) + B_{>k}(x)$. Moreover, we can choose $(\hat{U}, \hat{V}, \hat{\phi}) \in \hat{G}$, i.e. $\hat{U}|_0 = \mathbb{I}$, $\hat{V}|_0 = \mathbb{I}$ and $\hat{\phi}|_0 = Id$.

As both A and B are analytic matrices, these are the analytic equations. Therefore, by Artin's approximation theorem, [Artin1968], these equations have an *analytic* solution, $(U, V, \phi) \in G$, with $U|_0 = \mathbb{I}$, $V|_0 = \mathbb{I}$ and $\phi|_0 = Id$. Thus, $A(x)$ is analytically- G -equivalent to $A + B_{>k}$. Hence the analytic finite- G -determinacy.

2. Suppose $A \in \text{Mat}(m, n, C^\infty(\mathbb{R}^p, 0))$ is formally finitely determined, then, in particular, it is equivalent to a matrix whose expansion into the power series is *polynomial*. Thus, from now on, we assume each entry of A is a polynomial plus a flat function, so \hat{A} is a matrix of polynomials. As $\hat{A} \in \text{Mat}(m, n, \hat{R})$ is finitely- \hat{G} -determined, for any $B_{>k} \in \text{Mat}(m, n, R)$ the equation $\hat{U}\hat{A}(\hat{\phi})\hat{V} = \hat{A} + \hat{B}_{>k}$ has a formal solution.

Recall Borel's lemma, [Rudin-book, Chapter 19, ex.12]: any formal power series is presentable by a C^∞ function, i.e. for any $f \in \mathbb{R}[[x_1, \dots, x_p]]$ there exists $g \in C^\infty(\mathbb{R}^p, 0)$ such that $\hat{g} = f$. Thus the formal triple $(\hat{U}, \hat{V}, \hat{\phi}) \in \hat{G}$ is presentable by a smooth triple $(U, V, \phi) \in G$, satisfying:

$$(17) \quad UA(\phi)V - A - B_{>k} = a \text{ flat function.}$$

Present: $U = \hat{U} + u_0$, $V = \hat{V} + v_0$, $\phi = \hat{\phi} + \phi_0$, where u_0, v_0, ϕ_0 are flat (i.e. their expansion in power series are identically zero). We prove that these flat pieces can be adjusted so that the difference $(\hat{U} + u_0)A(\hat{\phi} + \phi_0)(\hat{V} + v_0) - A - B_{>k}$ is identically zero (and not just a flat function). For this we use the implicit function theorem. Expand the equation above in powers of u_0, v_0, ϕ_0 , then we get:

$$(18) \quad u_0A(x) + \phi_0\partial A(x) + A(x)v_0 + F_{\geq 2}(x, u_0, v_0, \phi_0) = a \text{ flat function.}$$

Here $F_{\geq 2}$ depends on all the ingredients, but it is at least of order 2 in the flat functions (u_0, v_0, ϕ_0) . Note that on the left we have precisely $\mathcal{A}h$, where h is the combined vector of u_0, v_0, ϕ_0 . We look for the solution of this equation, i.e. the vector h , in the form: $h = \mathcal{A}^T w$, where w is some flat function. Thus we should solve the equation: $\mathcal{A}\mathcal{A}^T w + F(x, \mathcal{A}^T w) = a$ flat function.

Note that $\mathcal{A}\mathcal{A}^T$ is a square matrix, whose inverse is: $\frac{(\mathcal{A}\mathcal{A}^T)^\vee}{\det(\mathcal{A}\mathcal{A}^T)}$, upstairs we have the matrix of co-factors. Further, as $F_{\geq 2}$ is of order at least two in $\mathcal{A}^T w$, we can expand it into Taylor series with the remainder:

$$(19) \quad F(\underline{x}, \underline{y}) = \underbrace{F(\underline{x}, 0)}_0 + \underbrace{\partial_{\underline{y}} F(\underline{x}, 0)\underline{y}}_0 + \left(\int_0^1 (1-t) \partial_{\underline{y}}^2 F(\underline{x}, t\underline{y}) dt \right) (\underline{y}, \underline{y})$$

Here the last integral gives a bilinear form, evaluated at \underline{y} . Therefore, for $\underline{y} = \mathcal{A}^T w$ we have:

$$(20) \quad w + \frac{(\mathcal{A}\mathcal{A}^T)^\vee}{\det(\mathcal{A}\mathcal{A}^T)} \left(\int_0^1 (1-t) \partial_{\underline{y}}^2 F(\underline{x}, t\mathcal{A}w) dt \right) (\mathcal{A}^T w, \mathcal{A}^T w) = \frac{\text{flat function}}{\det(\mathcal{A}\mathcal{A}^T)}$$

Note that $\mathcal{A}\mathcal{A}^T$ is the Gram-matrix of the rows of \mathcal{A} , [Gantmacher-book]. Therefore its determinant is non-negative, $\det(\mathcal{A}\mathcal{A}^T)|_{pt} \geq 0$, and vanishes iff the rows of $\mathcal{A}|_{pt}$ are linearly dependent, i.e. $\mathcal{A}|_{pt}$ has left kernel. But, as A is formally finitely determined, the matrix \hat{A} has no left kernel in the punctured neighborhood of the point. (Recall that by the initial assumption \hat{A} is a matrix of polynomials.) Therefore $\det(\mathcal{A}\mathcal{A}^T)|_{pt} > 0$ in the punctured neighborhood of the origin and vanishes precisely at the origin. But then, by Lojasiewicz inequality we get: $|\det(\mathcal{A}\mathcal{A}^T)| \geq C\|x\|^\delta$, for some $\delta \geq 0$. Therefore $\frac{\text{flat function}}{\det(\mathcal{A}\mathcal{A}^T)}$ is again a flat function and $\frac{w}{\det(\mathcal{A}\mathcal{A}^T)}$ is a flat function too. So, we are looking for the solution of the initial equation in the form $\det(\mathcal{A}\mathcal{A}^T)\mathcal{A}^T w'$, where w' is a flat function, satisfying:

$$(21) \quad w' + (\mathcal{A}\mathcal{A}^T)^\vee \left(\int_0^1 (1-t) \partial_{\underline{y}}^2 F(\underline{x}, t\mathcal{A}w) dt \right) (\mathcal{A}^T w', \mathcal{A}^T w') = \text{flat function}$$

And now we can invoke the implicit function theorem for w' , (note that the derivative of l.h.s. with respect to w' is non-zero), to get a smooth solution. Finally, by expanding this last equation into power series, we get: the obtained smooth solution is a flat function. ■

Remark 5.2. Instead of the space $\text{Mat}(m, n, R)$ we could choose any subspace, defined by a finite collection of algebraic equations, e.g. $\text{Mat}(m, n, \mathfrak{m})$, or (anti)symmetric matrices etc. Instead of the group \mathcal{G}_{lr} we could choose any subgroup of \mathcal{G}_{lr} which is defined by a finite number of algebraic equations (e.g. G_{lr} , G_{congr} , $\text{Aut}(R)$ etc). By direct check, the theorem holds with the obvious modifications in the proof.

5.2. An alternative proof of some statements.

Theorem 5.3. Let $A, B \in \text{Mat}(m, n, R)$, $n \geq m$ and $\dim(R) > 0$.

1. If A is finitely G_{lr} -determined then $I_m(A)$ contains a power of maximal ideal $\mathfrak{m} \subset R$. In other words, the support of the module $\text{Coker}(A)$ is a point.

2. If $I_1(B) \subset I_m(A)$ then $A + B \stackrel{G_r}{\sim} A$.

- 2'. In particular, if $I_m(A)$ contains a power of maximal ideal $\mathfrak{m} \subset R$, then A is finitely G_r -determined
3. If $n > m$ then $A \in \text{Mat}(m, n, R)$ is never finitely G_l -determined.

Proof. 1. We say that I_j, I_k eventually coincide if $I_j \cap \mathfrak{m}^N = I_k \cap \mathfrak{m}^N$ for $N \gg 0$. Note that I_1 contains \mathfrak{m}^N for $N \gg 0$. Otherwise, if $g \in \mathfrak{m}^N \setminus (\mathfrak{m}^N \cap I_1)$, then the matrix $A + g\mathbb{I}$ is not equivalent to A , as they have different I_1 's.

Suppose $I_m(A)$ does not contain \mathfrak{m}^N for any $N > 0$, so I_m and I_1 do not eventually coincide. Then exists $0 < k \leq (m-1)$ such that I_k contains \mathfrak{m}^N but I_{k+1} does not. So, for any $N > 0$ there exists an element in $\mathfrak{m}^{kN} \setminus (\mathfrak{m}^{kN} \cap I_{k+1})$ of the form $\prod_{i=1}^k f_i$ such that $f_i \in \mathfrak{m}^N \setminus (\mathfrak{m}^N \cap I_{k+1})$ for any i .

Consider the deformation $A + \epsilon B$, where $\epsilon \in \mathbb{K}$ is a parameter, B is "diagonal", i.e. $B_{ii} = f_i$ for $1 \leq i \leq k$ and $B_{ij} = 0$ otherwise. We prove that for some ϵ (and thus for generic ϵ) the matrix $A + \epsilon B$ is not G_{lr} equivalent to A , so A is not N -determined. It is enough to prove $I_k(A) \neq I_k(A + \epsilon B)$. Consider the $k \times k$ minor of $A + \epsilon B$ corresponding to the non-zero terms of B . If this minor belongs to $I_k(A)$, for any $\epsilon \in \mathbb{K}$, then $\prod f_i \in I_k(A)$. Contradicting the initial assumption.

2. Suppose $I_1(B) \subset I_m(A)$, i.e. the entries of B belong to $I_m(A)$. Let $\{g_j\}$ be the maximal minors of A , they generate $I_m(A)$. Thus one can present: $B = \sum g_j B_j$, where each B_j is a matrix of formal power series. We prove the equivalence $A \stackrel{G_r}{\sim} A + B$ in steps, after q 'th step all the (remaining) entries of B will belong to \mathfrak{m}^q . (Recall that for G_r the jet-by-jet-equivalence implies the ordinary equivalence.)

As $A|_0 = \mathbb{O}$, we get $B|_0 = \mathbb{O}$, so the 0'th step is completed. Suppose after the step $(q-1)$ some entries of B_1 are not in $\mathfrak{m}^{q-\text{order}(g_1)}$. Consider the matrix equation $A_{m \times n} C_{n \times n} = g_1(B_1)_{m \times n}$. We claim that there exists a solution to this, a matrix C of formal power series, vanishing at the origin. Indeed, present $g_1 \mathbb{I}_{m \times m} = A_1 A_1^\vee$, where A_1 is the maximal minor whose determinant is g_1 and A_1^\vee is the adjoint of A_1 . After a permutation of columns of A we can assume that A_1 is the left $m \times m$ submatrix of A . Then an immediate solution to the equation above is: $C_1 = \begin{pmatrix} A_1^\vee B_1 \\ \mathbb{O}_{(n-m) \times n} \end{pmatrix}$.

Therefore, if C_1 is any solution of $AC_1 = g_1 B_1$, consider $(A + \sum_{i \geq 1} g_i B_i)(\mathbb{I} - C_1) = A + \sum_{i \geq 2} g_i B_i - \sum_{i \geq 1} g_i B_i C_1$. Note that $(\mathbb{I} - C_1) \in G_r$ and $\sum_{i \geq 1} g_i B_i C_1 \in \text{Mat}(m, n, \mathfrak{m}^{q+1})$. (As we are interested in finite determinacy only, we can assume the order of entries of B_1 high enough, so that $C_1|_0 = \mathbb{O}$.) Continue similarly: for solutions $AC_j = g_j B_j$ apply the transformations $(A + \sum_{i \geq 1} g_i B_i)(\mathbb{I} - C_1)(\mathbb{I} - C_2) \dots (\mathbb{I} - C_k)$. Now:

$$(22) \quad \left((A + \sum_{i \geq 1} g_i B_i)(\mathbb{I} - C_1)(\mathbb{I} - C_2) \dots (\mathbb{I} - C_k) \right) - A \equiv \mathbb{O} \mod(\mathfrak{m}^{q+1}),$$

i.e. q 'th step is completed.

2'. Suppose $I_m(A) \supset \mathfrak{m}^N$, take any $B \in \text{Mat}(m, n, \mathfrak{m}^N)$, apply (2).

3. Suppose $A \in \text{Mat}(m, n, R)$ is finitely G_l -determined, then there exists a maximal minor of A whose determinant is not a zero divisor in R . By permutation of columns we can assume that A has the form $(A_1|A_2)$, where $A_1 \in \text{Mat}(m, m, R)$, $\det(A_1)$ is not a zero divisor and $A_2 \in \text{Mat}(m, n-m, R)$. Let $\mathbb{O} \neq B_2 \in \text{Mat}(m, n-m, R)$ then $A + B$ is never G_l -equivalent to A . Indeed, if $\phi(A_1|A_2) = (A_1|A_2 + B)$ then $\phi A_1 = A_1$. Multiply by the adjoint matrix, $A_1 A_1^\vee = \det(A_1) \mathbb{I}$ to get: $(\phi - \mathbb{I}) \det(A_1) = \mathbb{O}$. This forces $\phi = \mathbb{I}$. ■

REFERENCES

- [Arnol'd1971] V.I.Arnol'd, *Matrices depending on parameters*. (Russian) Uspehi Mat. Nauk 26 (1971), no. 2(158), 101–114.
- [Arnol'd-problems] V.I.Arnold, *Arnold's problems*. Translated and revised edition of the 2000 Russian original. Springer-Verlag, Berlin; PHSIS, Moscow, 2004. xvi+639 pp. ISBN: 3-540-20614-0
- [AGLV-1] V.I.Arnol'd, V.V.Goryunov, O.V.Lyashko, V.A.Vasil'ev, *Singularity theory. I*. Reprint of the original English edition from the series Encyclopaedia of Mathematical Sciences [Dynamical systems. VI, Encyclopaedia Math. Sci., 6, Springer, Berlin, 1993]. Springer-Verlag, Berlin, 1998. iv+245 pp. ISBN: 3-540-63711-7
- [AGLV-2] V.I.Arnol'd, V.A.Vasil'ev, V.V.Goryunov, O.V.Lyashko, *Singularities. II. Classification and applications*. (Russian) With the collaboration of B. Z. Shapiro. Itogi Nauki i Tekhniki, Current problems in mathematics. Fundamental directions, Vol. 39, 5256, Akad. Nauk SSSR, 1989.
- [Artin1968] M.Artin, *On the solutions of analytic equations*. Invent. Math. 5 1968 277–291
- [Birkhoff-1913] G.Birkhoff, *A theorem on matrices of analytic functions*. Math. Ann. 74 (1913), no. 3, 461
- [Bruce-Goryunov-Zakalyukin2002] J.W.Bruce, V.V.Goryunov, V.M.Zakalyukin, *Sectional singularities and geometry of families of planar quadratic forms*. Trends in singularities, 83–97, Trends Math., Birkhäuser, Basel, 2002
- [Bruce03] J.W.Bruce, *On families of symmetric matrices*. Dedicated to Vladimir I. Arnold on the occasion of his 65th birthday. Mosc. Math. J. 3 (2003), no. 2, 335360, 741
- [Bruce-Tari2004] J.W.Bruce, F.Tari, *On families of square matrices*. Proc. London Math. Soc. (3) 89 (2004), no. 3, 738–762.
- [Christensen-Sather-Wagstaff] L.W.Christensen, S.Sather-Wagstaff, *Descent via Koszul extensions*. J. Algebra 322 (2009), no. 9, 3026–3046
- [Dieudonné1949] J.Dieudonné, *Sur une généralisation du groupe orthogonal à quatre variables*. Arch. Math. 1, (1949). 282–287
- [Eisenbud-book] D.Eisenbud, *Commutative algebra. With a view toward algebraic geometry*. Graduate Texts in Mathematics, 150. Springer-Verlag, New York, 1995. xvi+785
- [Elkik73] R.Elkik, *Solutions d'équations à coefficients dans un anneau hensélien*. Ann. Sci. cole Norm. Sup. (4) 6 (1973), 553–603 (1974)
- [FSWW08] A.J.Frankild, S.Sather-Wagstaff, R.Wiegand, *Ascent of module structures, vanishing of Ext, and extended modules*. Special volume in honor of Melvin Hochster. Michigan Math. J. 57 (2008), 321337
- [Gantmacher-book] F.R.Gantmacher, *The theory of matrices*. Vols. 1, 2. Translated by K. A. Hirsch Chelsea Publishing Co., New York 1959 Vol. 1, x+374 pp. Vol. 2, ix+276 pp

- [Goryunov-Mond05] V.Goryunov, D.Mond, *Tjurina and Milnor numbers of matrix singularities*. J. London Math. Soc. (2) 72 (2005), no. 1, 205–224.
- [Goryunov-Zakalyukin03] V.V.Goryunov, V.M.Zakalyukin, *Simple symmetric matrix singularities and the subgroups of Weyl groups A_μ , D_μ , E_μ* . Dedicated to Vladimir I. Arnold on the occasion of his 65th birthday. Mosc. Math. J. 3 (2003), no. 2, 507–530, 743–744
- [Grothendieck-1957] A.Grothendieck, *Sur la classification des fibrés holomorphes sur la sphère de Riemann*. Amer. J. Math. 79 (1957), 121–138
- [Keller-Murfet-Van den Bergh2008] B.Keller, D.Murfet, M.Van den Bergh *On two examples by Iyama and Yoshino*, Compos. Math. 147 (2011), no. 2, 591–612.
- [Leuschke-Wiegand] G.J.Leuschke, R.Wiegand, *Cohen-Macaulay representations*. Mathematical Surveys and Monographs, 181. American Mathematical Society, Providence, RI, 2012. xviii+367 pp.
- [Looijenga-book] E. Looijenga, *Isolated Singular Points on Complete Intersections*. London Math. Soc. LNS 77, CUP, 1984.
- [Mather1970] J.N.Mather, *Stability of C^∞ mappings. VI: The nice dimensions*. Proceedings of Liverpool Singularities-Symposium, I (1969/70), pp. 207–253. Lecture Notes in Math., Vol. 192, Springer, Berlin, 1971
- [Mather1973] J.N.Mather, *Generic projections*. Ann. of Math. (2) 98 (1973), 226–245
- [Molnár2007] L.Molnár, *Selected preserver problems on algebraic structures of linear operators and on function spaces*. Lecture Notes in Mathematics, 1895. Springer-Verlag, Berlin, 2007.
- [du Plessis-Wall] A.du Plessis, C.T.C.Wall, *The geometry of topological stability*. London Mathematical Society Monographs. New Series, 9. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995
- [Rudin-book] W.Rudin, *Real and complex analysis*. Third edition. McGraw-Hill Book Co., New York, 1987. xiv+416 pp
- [Tougeron1968] J.C.Tougeron, *Idéaux de fonctions différentiables. I*. Ann. Inst. Fourier (Grenoble) 18 1968 fasc. 1, 177–240
- [Yoshino-book] Y.Yoshino, *Cohen-Macaulay modules over Cohen-Macaulay rings*. London Mathematical Society Lecture Note Series, 146. Cambridge University Press, Cambridge, 1990. viii+177 pp.
- [Wall-1981] C.T.C.Wall, *Finite determinacy of smooth map-germs*. Bull. London Math. Soc. 13 (1981), no. 6, 481–539.

DEPARTMENT OF MATHEMATICS, BEN GURION UNIVERSITY OF THE NEGEV, P.O.B. 653, BE'ER SHEVA 84105, ISRAEL.
E-mail address: `genrich@math.bgu.ac.il`

DEPARTMENT OF MATHEMATICS, BEN GURION UNIVERSITY OF THE NEGEV, P.O.B. 653, BE'ER SHEVA 84105, ISRAEL.
E-mail address: `dmitry.kerner@gmail.com`